MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

MULTIPLE SIGNAL DIRECTION FINDING WITH THINNED LINEAR ARRAYS

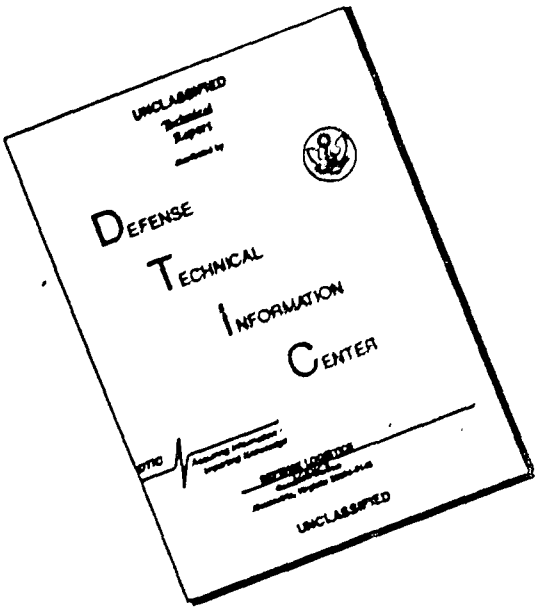
D.F. DELONG
Group 44

PROJECT REPORT TST-68

15 APRIL 1983

LEXINGTON MASSACHUSETTS

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

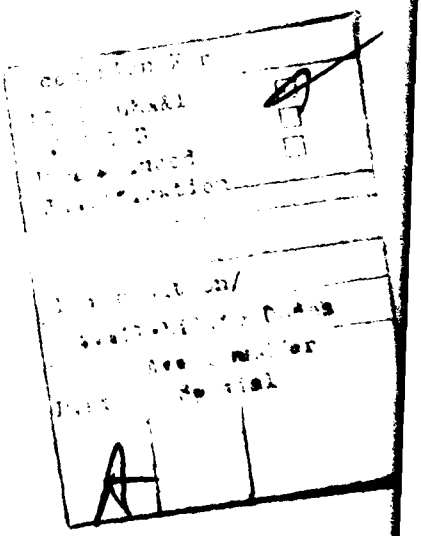
ABSTRACT

The evaluation of the direction-finding capabilities of non-uniform arrays is approached via algorithm-independent lower bounds on achievable angle estimation errors. Two classes of bounds are considered. The first, known as the Cramer-Rao bound, applies only to unbiased estimates. Compact analytical expressions for these bounds are developed which are applicable to very general direction-finding problems, including an arbitrary number of emitters.

It is well-known that Cramer-Rao bounds are overly optimistic at low signal-to-noise ratios. As this ratio decreases, a point is reached at which estimation accuracy decreases abruptly. Another class of bounds, known as Ziv-Zakai bounds, provide information about the location of this threshold point. Their study suggests that poor direction-finding performance occurs in situations where the emitter direction vectors are part of a set which is nearly linearly dependent.

Such linear dependence does not occur in the case of uniformly spaced linear arrays (without getting lobes). However, it does occur when elements are removed from such arrays. A systematic test is developed to test a given array geometry for this condition.

Finally, direction-finding performance of the QUICK LOOK array is evaluated via both bounds and simulation.



CONTENTS

CONTENTS (cont'd.)

LIST OF ILLUSTRATIONS

I. INTRODUCTION

The evaluation of the direction-finding capabilities of non-uniform arrays is approached via algorithm-independent lower bounds on achievable angle estimation errors. Two classes of bounds are considered. The first, known as the Cramer-Rao bound, applies only to unbiased estimates. Since almost all "reasonable" estimates are asymptotically unbiased as the signal-to-noise ratio becomes large, these bounds are useful for evaluating performance in that situation. Compact analytical expressions for these bounds are developed which are applicable to very general direction-finding problems, including an arbitrary number of emitters.

It is well-known that Cramer-Rao bounds are overly optimistic at low signal-to-noise ratios. As this ratio decreases, a point is reached at which estimation accuracy decreases abruptly. Another class of bounds, known as Ziv-Zakai bounds, provide information about the location of this threshold point. Calculation of these bounds requires numerical maximization of a function. In the case of more than one emitter, this function is multidimensional and the computational load appears to be prohibitive. Despite this difficulty, the study of Ziv-Zakai bounds suggests that poor threshold behavior occurs in situations where the emitter direction vectors are part of a set which is nearly linearly dependent.

The simplest example of linear dependence is a grating lobe of a uniformly spaced linear array. When the array elements are separated by more than half a wavelength ($\lambda/2$), there exist two (or more) directions which produce the same direction vector. It is impossible to determine from the array data whether the signal is from direction A, direction B, or both. When the spacing is less than $\lambda/2$, it can be shown that for every set of distinct directions, the corresponding set of direction vectors is linearly independent, and therefore no ambiguity problems exist.

A special case of non-uniformly spaced linear arrays, called thinned linear arrays, are those obtained by removing elements from a uniform linear array. These exhibit ambiguities of a more complicated nature. For example, when a set of three direction vectors is linearly dependent, the observed data

could have come from direction A and B, or A and C, or B and C, or all three. A given array geometry can be systematically tested for the presence of linearly dependent direction vectors.

Finally, direction-finding performance of the QUICK LOOK array is evaluated via both bounds and simulation. Results show that the currently used "phase only" processing is essentially optimum for a single emitter. For two emitters, a processing algorithm called MUSIC provides nearly optimum performance except in certain difficult cases. Further work is needed to determine whether these failures are due to the array geometry or to a weakness in the algorithm itself.

II. PERFORMANCE BOUNDS

The first question that arises in the study of unequally spaced linear arrays is that of array geometry. What sort of array geometries are "best" for locating multiple emitters? Evaluation of an array via simulation requires the choice of an algorithm; if performance is bad, it may be due to a poor choice of algorithm. The study of algorithm - independent performance bounds is motivated by the desire to separate these two factors.

The oldest and best known bound is the Cramer-Rao bound. It is valid only for unbiased estimators (or estimators of known bias). When the observations from which an estimate is to be made are of the form

$$\underline{z} = A\underline{s}(\underline{\alpha}) + \underline{\xi}$$

where $\underline{s}$ is a deterministic vector which depends on a set of unknown parameters $\underline{\alpha}$, and $\underline{\xi}$ is a Gaussian noise vector, it can be shown that there exists an estimator which achieves the Cramer-Rao bound asymptotically as the signal-to-noise ratio $A^2/E|\xi_1|^2$ becomes large.

As signal-to-noise ratio decreases, a point is reached at which the accuracy of the estimate degrades abruptly. This value of S/N is referred to as the threshold S/N; in general, it will depend on the estimation algorithm. The Cramer-Rao bound gives no information about this behavior. A bound which does give such information is the Ziv-Zakai bound.

1. The Cramer-Rao Bound

A. General Discussion

Derivations of the multiparameter Cramer-Rao bound are readily available in the literature (e.g., [1]). The result may be stated as follows: let $\underline{\alpha}$ be a real K-dimensional vector parameter, and let $p(\underline{z}/\underline{\alpha})$ be a probabilistic mapping from this parameter space into an observation space whose elements are N-dimensional real or complex vectors $\underline{z}$. The Cramer-Rao bound states that the covariance matrix of any estimate $\hat{\underline{\alpha}}$ of $\underline{\alpha}$ satisfies

$$\mathrm{Cov}(\underline{\alpha}) - \left(\frac{\partial \underline{m}}{\partial \underline{\alpha}}\right)^T F^{-1} \frac{\partial \underline{m}}{\partial \underline{\alpha}} \geq 0 \qquad (2.1)$$

i.e. the difference of these two matrices is positive semidefinite. Here

$$\underline{m} = E(\underline{\alpha})$$

$$\frac{\partial \underline{m}}{\partial \underline{\alpha}} = \left[\frac{\partial m_i}{\partial \alpha_j}\right]$$

and $$F = E\left\{\left(\frac{\partial \ln p}{\partial \underline{\alpha}}\right)^T \frac{\partial \ln p}{\partial \underline{\alpha}}\right\}$$

is called the Fisher information matrix.

For unbiased estimates, $\frac{\partial \underline{m}}{\partial \underline{\alpha}} = I$, the identity matrix, and the bound becomes simply

$$\mathrm{Cov}(\hat{\underline{\alpha}}) \geq F^{-1} \qquad (2.2)$$

The remainder of this section will discuss only unbiased estimates.

It is frequently the case that only some of the unknown parameters, denoted by β, are of interest. The remaining parameters γ, called "nuisance" parameters, are not of interest but must be estimated. The information matrix can then be partitioned as follows:

$$F = \begin{bmatrix} F_{\beta\beta} & F_{\beta\gamma} \\ F_{\gamma\beta} & F_{\gamma\gamma} \end{bmatrix}$$

It is readily shown that

$$F^{-1} = \begin{bmatrix} \left(F_{\beta\beta} - F_{\beta\gamma} F_{\gamma\gamma}^{-1} F_{\gamma\beta}\right)^{-1} & * \\ * & * \end{bmatrix} \qquad (2.3)$$

Expressions for the other submatrices are easily obtained, but will not be required here. Since $\mathrm{Cov}(\hat{\underline{\alpha}}) \geq F^{-1}$, it follows that

$$\mathrm{Cov}(\hat{\underline{\beta}}) \geq \left(F_{\beta\beta} - F_{\beta\gamma} F_{\gamma\gamma}^{-1} F_{\gamma\beta}\right)^{-1} \qquad (2.4)$$

for otherwise a principal minor of Cov $(\alpha) - F^{-1}$ would be negative.

The matrix whose inverse appears on the right side of (2.4) is often called the reduced information matrix for the parameter set β, denoted by $F^{(-)}_{\beta\beta}$. It consists of the appropriate submatrix of F reduced by a positive semidefinite matrix.

Application of this result to the case of a single parameter shows that the variance in the estimate of a single parameter must exceed the inverse of the corresponding diagonal element of F, i.e.,

$$\mathrm{Cov}(\hat{\underline{\beta}}) \geq \frac{1}{F_{\beta\beta}} \tag{2.5}$$

B. Estimation of Signal Parameters in Gaussian Noise

An important special case to which the Cramer-Rao bound may be applied is the estimation of the parameters of a deterministic signal observed in the presence of additive Gaussian noise, i.e.,

$$\underline{z} = \underline{s}(\underline{\alpha}) + \underline{\xi}$$

where $\underline{\xi}$ is a circular complex Gaussian vector with zero mean and covariance Λ.

The logarithm of the likelihood function is then of the form

$$\ell n\, p(\underline{z}/\underline{\alpha}) = -M\ell m - \ell n|\Lambda| - (\underline{z} - \underline{s}(\underline{\alpha}))^H \Lambda^{-1}(\underline{z}-\underline{s}(\underline{\alpha}))$$

with derivative

$$\frac{\partial\, \ell n\, p}{\partial\, \underline{\alpha}} = 2\,\mathrm{Re}\left\{(\underline{z}-\underline{s})^H \Lambda^{-1} \frac{\partial \underline{s}}{\partial \underline{\alpha}}\right\} = 2\mathrm{Re}\left\{\xi^H \Lambda^{-1} \frac{\partial \underline{s}}{\partial \underline{\alpha}}\right\}$$

The Fisher information matrix is then

$$F = E\left\{\left(\frac{\partial\, \ell n p}{\partial \underline{\alpha}}\right)^T \frac{\partial\, \ell n p}{\partial \underline{\alpha}}\right\} = 2\,\mathrm{Re}\left\{\left(\frac{\partial \underline{s}}{\partial \underline{\alpha}}\right)^H \Lambda^{-1} \frac{\partial \underline{s}}{\partial \underline{\alpha}}\right\} \tag{2.6}$$

It is often the case that some of the real parameters $\underline{\alpha}$ are actually the real and imaginary parts of complex-valued parameters, e.g. unknown complex amplitude of the signal vector $\underline{s}$. Denote this set of complex parameters by the vector $\underline{c}$ and assume that $\underline{s}(\underline{c})$ is an analytic vector function of $\underline{c}$ i.e., each component of $\underline{s}$ is an analytic function of each of the components of $\underline{c}$.

Define $C = \Lambda^{-1/2}\frac{\partial \underline{s}}{\partial \underline{c}}$. It follows from the Cauchy-Riemann equations that*

$$\Lambda^{-1/2}\frac{\partial \underline{s}}{\partial \mathrm{Re}\,\underline{c}} = C, \quad \Lambda^{-1/2}\frac{\partial \underline{s}}{\partial \mathrm{Im}\,\underline{c}} = jC$$

Let $\underline{u}$ now denote the remaining real parameters and define $A = \frac{\partial \underline{s}}{\partial \underline{u}}$. The Fisher information matrix for the (transposed) parameter vector $(\underline{u}^T, \mathrm{Re}\,\underline{c}^T, \mathrm{Im}\,\underline{c}^T)$ is then

$$F = 2\,\mathrm{Re}\begin{bmatrix} A^HA & A^HC & jA^HC \\ & C^HC & jC^HC \\ \text{Hermitian} & & C^HC \end{bmatrix}$$

The reduced information matrix for the parameter vector $\underline{u}$ alone is then,

$$F_{uu}^{(-)} = 2\{\mathrm{Re}\,A^HA - (\mathrm{Re}\,A^HC, -\mathrm{Im}A^HC)\begin{bmatrix} \mathrm{Re}\,C^HC & -\mathrm{Im}\,C^HC \\ \mathrm{Im}\,C^HC & \mathrm{Re}\,C^HC \end{bmatrix}\begin{pmatrix} \mathrm{Re} & C^HA \\ \mathrm{Im} & C^HA \end{pmatrix}\}$$

Evaluation of this expression is greatly simplified by making use of the isomorphism between multiplication of complex NxN matrices and of real 2Nx2N matrices of special form.

$$\begin{bmatrix} \mathrm{Re}\,A & -\mathrm{Im}\,A \\ \mathrm{Im}\,A & \mathrm{Re}\,A \end{bmatrix}_A \cdot \begin{bmatrix} \mathrm{Re}\,B & -\mathrm{Im}\,B \\ \mathrm{Im}\,B & \mathrm{Re}\,B \end{bmatrix}_B = \begin{bmatrix} \mathrm{Re}\,AB & -\mathrm{Im}\,AB \\ \mathrm{Im}\,AB & \mathrm{Re}\,AB \end{bmatrix}_{AB}$$

---

*We use the letter j to denote $\sqrt{-1}$ throughout.

It follows immediately that

$$\begin{bmatrix} \text{Re } B & -\text{Im } B \\ \text{Im } B & \text{Re } B \end{bmatrix}^{-1} = \begin{bmatrix} \text{Re}(B^{-1}) & -\text{Im}(B^{-1}) \\ \text{Im}(B^{-1}) & \text{Re}(B^{-1}) \end{bmatrix}$$

and that

$$(\text{Re } A, -\text{Im}A) \begin{bmatrix} \text{Re } B & -\text{Im } B \\ \text{Im } B & \text{ReB} \end{bmatrix}^{-1} \begin{bmatrix} \text{Re } C \\ \text{Im } C \end{bmatrix} = \text{Re}(AB^{-1}C)$$

Application of these results to the above expression yields

$$F_{uu}^{(-)} = \text{Re}\{A^H(I - C(C^HC)^{-1}C^H)A\} \tag{2.7}$$

and the Cramer-Rao bound on the covariance matrix of any unbiased estimate $\hat{\underline{u}}$ of $\underline{u}$ is

$$\text{Cov}(\hat{\underline{u}}) \geq [F_{uu}^{(-)}]^{-1} \tag{2.8}$$

C. Multiple Elementary Signals in Gaussian Noise

This is actually a special case of the preceding section. The observation is modelled as

$$\underline{z} = \sum_{k=1}^{K} p_k \underline{v}(u_k) + \underline{\xi} = V(\underline{u})\,\underline{p} + \underline{\xi}$$

where

$$\underline{v}^T(u) = \frac{1}{\sqrt{M}}(e^{j2\pi ux_1/\lambda}, e^{j2\pi ux_2/\lambda}, \ldots e^{j2\pi ux_M/\lambda})$$

is a normalized direction vector which represents the response of an ideal linear array with elements at positions x_m to a plane wave from direction u.

It is convenient to define the origin of the x coordinate so that $\sum x_m = 0$. The "elementary" signals are all of the form $\underline{v}(u)$ and have unknown complex amplitudes. The results of the last section are directly applicable with the substitution

$$\underline{s}\ (\underline{\alpha}) = V(\underline{u})\underline{p}$$

Thus,

$$A = \Lambda^{-1/2}\ \frac{\partial \underline{s}}{\partial \underline{u}} = \Lambda^{-1/2}\,\dot{V}\,[\underline{p}]$$

$$C = \Lambda^{-1/2}\frac{\partial \underline{s}}{\partial \underline{Rep}} = \Lambda^{-1/2}\,V$$

where

$$\dot{V} = \frac{\partial V}{\partial u} = \left[\ \frac{\partial \underline{v}}{\partial u_1}\ \frac{\partial \underline{v}}{\partial u_2}\ \cdots\ \frac{\partial \underline{v}}{\partial u_k}\ \right]$$

and

$$[\underline{p}] = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_K \end{bmatrix}.$$

The reduced information matrix for $\underline{u}$ is then obtained using (2.4). The result is

$$F_{uu}^{(-)} = 2\ \mathrm{Re}\ \{[\underline{p}]^H\ (\ddot{W} - \dot{W}^H W^{-1} \dot{W})\ [\underline{p}]\} \qquad (2.9)$$

where

$$W \triangleq V^H\,\Lambda^{-1} V \qquad (2.10a)$$

$$\dot{W} \triangleq V^H\,\Lambda^{-1}\dot{V} \qquad (2.10b)$$

$$\ddot{W} \triangleq \dot{V}^H\,\Lambda^{-1}\,\dot{V} \qquad (2.10c)$$

The $\dot{W}$ notation is mnemonic only and should not be confused with that for derivatives.

The Cramer-Rao bound on the covariance matrix of any unbiased estimate of $\underline{u}$ is then given by

$$\mathrm{Cov}(\hat{\underline{u}}) \geq \frac{1}{2}\ \mathrm{Re}^{-1}\ \{[\underline{p}]^H(\tilde{W} - \dot{W}^H W^{-1}\dot{W})[\underline{p}]\} \qquad (2.11)$$

D. Extension to Multiple Observations

In the multiple look case, the question of signal coherence arises. Two limiting cases are of particular interest. In the incoherent case, the signal amplitudes are completely independent from one look to the next; they are treated as additional unknown parameters. The n^{th} observation is modelled as

$$\underline{z}_n = V(\underline{u})\underline{p}_n + \underline{\xi}_n \qquad n=1,..N$$

The most concise derivation of the information matrix for this case makes use of the Kronecker product [2] of two matrices, defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & a_{1N}B \\ a_{21}B & & \\ a_{M1}B & & a_{MN}B \end{bmatrix}$$

If A is M X N and B is R X S, their Kronecker product is MR X NS. The following easily verified properties of the Kronecker product will be required.

$$(A \otimes B)^H = A^H \otimes B^H$$

$$(A \otimes B)\ (C \otimes D) = AC \otimes BD \qquad \text{if A,C and B,D are conformable}$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$$\underline{a} \otimes \underline{b}^H = \underline{a}\ \underline{b}^H$$

Following the development of the preceding section

$$\underline{s}_n(\alpha) = V(\underline{u})\underline{p}_n$$

$$A_n = \frac{\partial \underline{s}_n}{\partial \underline{u}} = \Lambda^{-1/2}\ \dot{V}\ |\underline{p}_n|$$

$$C_{mn} = \frac{\partial \underline{s}_m}{\partial \mathrm{Re}\,\underline{p}_n} = \delta_{mn}\Lambda^{-1/2}V$$

Define the concatenated vectors

$$\underline{s}^T = (\underline{s}_1^T,\ \underline{s}_2^T,\ \ldots \underline{s}_N^T)$$

$$\underline{p}^T = (\underline{p}_1^T,\ \underline{p}_2^T,\ \ldots \underline{p}_N^T)$$

Now

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} = \begin{bmatrix} \Lambda^{-1/2}\ \dot{V}\ [\underline{p}_1] \\ \Lambda^{-1/2}\ \dot{V}\ [\underline{p}_2] \\ \vdots \\ \Lambda^{-1/2}\ \dot{V}\ [\underline{p}_N] \end{bmatrix} = (I \otimes \Lambda^{-1/2}\ \dot{V})Q \qquad (2.12)$$

where

$$Q \triangleq \begin{bmatrix} [\ \underline{p}_1\ ] \\ [\ \underline{p}_2\ ] \\ \vdots \\ [\ \underline{p}_N\ ] \end{bmatrix}$$

and

$$C = \begin{bmatrix} C_{11} & & 0 \\ & C_{22} & \\ & & \ddots \\ 0 & & C_{NN} \end{bmatrix} = \begin{bmatrix} \Lambda^{-1/2} V & & \\ & \Lambda^{-1/2} V & \\ & & \ddots \Lambda^{-1/2} V \end{bmatrix} = I \otimes \Lambda^{-1/2} V \qquad (2.13)$$

From (2.12) and (2.13) there follows, again using the notation of (2.10)

$$A^H A = Q^H (I \otimes \Lambda^{-1/2} \dot{V})^H \ (I \otimes \Lambda^{-1/2} \dot{V}) Q = Q^H (I \otimes \ddot{W}) Q \qquad (2.14a)$$

$$C^H A = (I \otimes \Lambda^{-1/2} V^H) \ (I \otimes \Lambda^{-1/2} \dot{V}) Q = (I \otimes \dot{W}) Q \qquad (2.14b)$$

$$(C^H C) = (I \otimes \Lambda^{-1/2} V)^H \ (I \otimes \Lambda^{-1/2} V) = (I \otimes W) \qquad (2.14c)$$

Use of (2.14) in (2.7) gives

$$F_{uu}^{(-)} = 2 \, \mathrm{Re}\{ Q^H [I \otimes \ddot{W} - (I \otimes \dot{W})^H (I \otimes W^{-1}) (I \otimes \dot{W})] Q \}$$

$$= 2 \, \mathrm{Re}\{ \sum_{n=1}^{N} [\underline{p}_n]^H (\ddot{W} - \dot{W}^H W^{-1} \dot{W}) [\underline{p}_n] \} \qquad (2.15)$$

which is the reduced information matrix for the multiple-look incoherent case.

In the other limiting case of complete coherence, the amplitude and phase relationships among the signals are fixed from look to look; only the overall complex amplitude b_n changes. Thus the n^{th} vector observation is modeled as

$$\underline{z}_n = b_n V(\underline{u}) \underline{p} + \underline{\xi}_n \qquad n=1,N; b_1=1$$

The number of unknown parameters in this model is 2(N-1)+K+2K, as opposed to 2NK+K in the incoherent case.

Proceeding as before

$$\underline{s}_n(\underline{u}) = b_n V(\underline{u}) \underline{p}$$

$$A_n = \Lambda^{-1/2} \frac{\partial s_n}{\partial \underline{u}} = b_n \Lambda^{-1/2} \dot{V}[\underline{p}]$$

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} = \begin{bmatrix} \Lambda^{-1/2} \dot{V}\ [\underline{p}] \\ b_2 \Lambda^{-1/2} \dot{V}\ [\underline{p}] \\ \vdots \\ b_N \Lambda^{-1/2} \dot{V}\ [\underline{p}] \end{bmatrix} = \underline{b}_o \otimes \Lambda^{-1/2} \dot{V}[\underline{p}]$$

where $\underline{b}_0^T = (1, b_2, \ldots b_N) = (1, \underline{b}^T)$.

Combine all the complex parameters into a vector $\underline{c}^T = (\underline{p}^T, \underline{b}^T)$. Then

$$\frac{\partial s_n}{\partial \mathrm{Re}\underline{p}} = b_n V$$

$$\frac{\partial s_n}{\partial \mathrm{Re}b_m} = \delta_{nm} V\underline{p} \qquad n=1, N;\ m=2, N$$

$$C_n = \Lambda^{-1/2} \left[\frac{\partial s_n}{\partial \mathrm{Re}\underline{p}} , \frac{\partial s_n}{\partial \mathrm{Re}\underline{b}} \right]$$

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} = \begin{bmatrix} \Lambda^{-1/2} V & 0 & 0 \\ b_2 \Lambda^{-1/2} V & \Lambda^{-1/2} V\underline{p} & \\ \vdots & & \ddots \\ b_N \Lambda^{-1/2} V & 0 & \Lambda^{-1/2}\ V\underline{p} \end{bmatrix} = [\underline{b}_o \otimes \Lambda^{-1/2} V,\ I_o \otimes \Lambda^{-1/2} V\underline{p}]$$

where $I_0^T = [\underline{0} \mid I_{N-1}]$.

Then

$$A^H A = \underline{b}_0^H \underline{b}_0 \, \Phi(\underline{p})^H \ddot{W}(\underline{p}) = d^2 |\underline{p}|^H \ddot{W}(\underline{p})$$

where

$$d^2 = \underline{b}_0^H \underline{b}_0 = 1 + \sum_{n=2}^{N} |b_n|^2, \text{ and}$$

$$C^H A = \begin{bmatrix} \underline{b}_0^H \Phi V^H \Lambda^{-1/2} \\ I_0^H \Phi(\underline{p})^H V^H \Lambda^{-1/2} \end{bmatrix} \left[\underline{b}_0 \Phi \Lambda^{-1/2}, \; V(\underline{p}) \right] = \begin{bmatrix} d^2 I_L \\ \underline{b} \, \underline{p}^H \end{bmatrix} \dot{W}(\underline{p})$$

$$C^H C = \begin{bmatrix} \underline{b}_0^H \Phi V^H \Lambda^{-1/2} \\ I_0^H \Phi \underline{p}^H V^H \Lambda^{-1/2} \end{bmatrix} \left[\underline{b} \Phi \Lambda^{-1/2} V, \; I_0 \Phi \Lambda^{-1/2} V \underline{p} \right] = \begin{bmatrix} d^2 W & W \underline{p} \, \underline{b}^H \\ \underline{b} \, \underline{p}^H W & \underline{p}^H W \underline{p} I_{N-1} \end{bmatrix}$$

Since $C^H C \begin{bmatrix} W^{-1} \\ 0 \end{bmatrix} = \begin{bmatrix} d^2 I_L \\ \underline{b} \, \underline{p}^H \end{bmatrix}$

there follows immediately

$$A^H C (C^H C)^{-1} C^H A = A^H C \, (C^H C)^{-1} \begin{bmatrix} d^2 I_L \\ \underline{b} \, \underline{p}^H \end{bmatrix} \dot{W}(\underline{p})$$

$$= |\underline{p}|^H \Phi^H \left[d^2 I \; \underline{p} \, \underline{b}^H \right] \begin{bmatrix} W^{-1} \\ 0 \end{bmatrix} \dot{W}(\underline{p})$$

$$= d^2 |\underline{p}|^H \Phi^H W^{-1} \dot{W}(\underline{p})$$

and finally, from (2.7)

$$F_{uu}^{(-)} = 2\, d^2 \mathrm{Re}\{|\underline{p}|^H(\ddot{W} - \dot{W}^H W^{-1}\dot{W})|\underline{p}|\} \qquad (2.16)$$

which is the reduced information matrix for the multiple-look coherent case. The Cramer-Rao bound on the error in estimating $\underline{u}$ is again the inverse of this matrix.

Although the derivation in the two cases is quite different, it is a curious fact that (2.16) can be obtained formally simply by making the substitution $\underline{p}_n = b_n \underline{p}$ in (2.15).

The principle results of this section are equations (2.15) and (2.16), which are the reduced information matrices for estimating the directions of arrival of multiple emitters in interference of an arbitrary nature, from multiple independent observations. Equation (2.15) is for the incoherent case in which the unknown complex amplitudes of the emitters change independently from observation to observation. Equation (2.16) covers the completely coherent case in which the unknown complex amplitudes have the same fixed relationship on every observation; the only fluctuation is a complex scale factor which changes from one observation to the next.

E. Examples

Single Emitter in White Noise

For a single emitter in white noise with covariance $\sigma^2 I$, equations (2.10) become simply $W = 1$, $\dot{W} = 0$, $\ddot{W} = \frac{1}{M} \sum (\frac{2\pi x_m}{\lambda})^2 = (2\pi \ell)^2$. The Cramer-Rao bound (2.11) is then simply

$$\mathrm{Cov}\,(\hat{u}) \geq \frac{\sigma^2}{2 \sum_{n=1}^{N} |p|^2} \frac{1}{(2\pi \ell)^2} \qquad (2.17)$$

The covariance of u is seen to be inversely proportional to signal-to-noise ratio, independent of the true direction u, and dependent on array geometry only through the "RMS array length" ℓ.

Two Emitters in White Noise

Bounds for the case of two emitters with directions u_1, u_2 are more

complicated. Results have been obtained for a sequence of cases of increasing complexity. The amplitude, phase, and direction of the first emitter are assumed to be unknown parameters. The corresponding parameters for the second interfering emitter are known or unknown, as specified.

(1) Interference Known

If all of the parameters of the second emitter are known, it can be subtracted out and does not degrade the accuracy. Thus we get the single emitter bound.

$$\text{Cov}\,(\hat{u}_1) > \frac{\sigma^2}{2\sum |p_{n1}|^2}\,\frac{1}{(2\pi\ell)^2} \triangleq \sigma_1^2$$

(2) Interference Amplitude and Phase Unknown

$$\text{Cov}(\hat{u}_1) > \sigma_1^2 \quad \frac{}{1-\dfrac{\dfrac{1}{(2\pi\ell)^2}\,|\underline{v}_I^H\,\dot{\underline{v}}_s|^2}{(1-|\underline{v}_I^H\,\underline{v}_s|^2)}} \triangleq \frac{\sigma_1^2}{\eta} \qquad (2.18)$$

The function $\eta(u_s, u_I)$ is an efficiency factor ($0<\eta<1$) which depends on the source directions and the array geometry.

(3) Interference Amplitude, Phase, and Direction Unknown

$$\text{Cov}(\hat{u}_1) > \frac{\sigma_1^2}{\eta}\,\frac{1}{1-\text{Re}^2\{\dfrac{1}{(2\pi\ell)^2\,\eta}\,(\dot{\underline{v}}_I^H\dot{\underline{v}}_I + \dfrac{\underline{v}_s^H\,\underline{v}_I(\underline{v}_I^H\dot{\underline{v}}_s)^2}{1-|\underline{v}_s^H\,\underline{v}_I|^2})e^{j\phi}\}} \qquad (2.19)$$

$$\triangleq \sigma_1^2\,\frac{1}{\eta}\,\frac{1}{1-\text{Re}^2\{\gamma e^{j\phi}\}}$$

The last factor in the denominator, which has been called a correlation or coherency factor [3], is critically dependent on the phase difference between the signals. In the most favorable situation $\text{Re}(\gamma e^{j\phi}) = 0$ and no degradation is caused by not knowing the direction of the interferer. In the least

favorable circumstance $\mathrm{Re}\ \gamma e^{j\psi} = |\gamma|$ and the correlation factor is $1-|\gamma|^2$.

From these examples we conclude that the efficiency factor is due to the unknown strength of the interference, while the coherency factor is due to unknown direction.

When multiple observations are available, it is instructive to compare the reduced information matrices for the incoherent and coherent cases. For the incoherent case, we have from (2.15)

$$F_{uu}^{(-)} = \frac{2\eta(2\pi\ell)^2}{\sigma^2}\,\mathrm{Re}\sum_{n=1}^{N}[\underline{p}_n]^H\begin{bmatrix}1 & \xi^* \\ \xi & 1\end{bmatrix}[\underline{p}_n] \tag{2.20}$$

$$= \frac{2\eta(2\pi\ell)^2}{\sigma^2}\begin{bmatrix}\sum|p_{n1}|^2 & \mathrm{Re}\{\gamma\sum p_{n1}^* p_{n2}\} \\ \text{symmetric} & \sum|p_{n2}|^2\end{bmatrix}$$

The bound depends on the actual set of complex signal amplitudes which occurred. If the two signals are really incoherent, the off-diagonal term will be small and the errors will be almost independent.

The completely coherent case gives

$$F_{uu}^{(-)} = \frac{2\eta(2\pi\ell)^2}{\sigma^2}d^2\begin{bmatrix}|p_1|^2 & \mathrm{Re}\{\gamma p_1^* p_2\} \\ \text{symmetric} & |p_2|^2\end{bmatrix} \tag{2.21}$$

The coupling term here is not reduced by multiple observations, and so the resulting bound on the variance contains the correlation factor $1-\mathrm{Re}^2(\gamma e^{j\phi})$.

One Emitter in the Presence of One Interfering Emitter and White Noise

The amplitude, phase, and direction of the emitter are modelled as unknown parameters. The interference is modelled as a signal from a known

direction u_I with a complex Gaussian amplitude of known power. The covariance matrix of interference plus noise is thus

$$A = \sigma^2(I + \rho_I \underline{v}_I \underline{v}_I^H)$$

with inverse

$$A^{-1} = \frac{1}{\sigma^2}(I - \gamma \underline{v}_I \underline{v}_I^H)$$

where $\gamma = \frac{\rho_I}{\rho_I+1}$. After considerable algebra we find

$$\sigma_u^2 > \frac{\sigma^2}{2 \sum |pn|^2} \frac{1}{(2\pi\ell)^2} \frac{1}{1 - \frac{1}{(2\pi\ell)^2} \frac{\gamma |\underline{v}_I^H \underline{v}_\theta|^2}{1-\gamma |\underline{v}_I^H \underline{v}_\theta|^2}} \qquad (2.22)$$

This is the same as (2.18) except for the factor γ, which approaches 1 as the interference becomes large relative to the noise. Note also that as the interference power goes to zero, (2.22) becomes the one emitter bound (2.17). This bound will be compared later to a similar one in which the emitter signal is also modelled as a Gaussian process.

F. Cramer-Rao Bounds for the Gaussian Signal Model

The preceding paragraphs have presented Cramer-Rao bounds on the accuracy of angle of arrival estimates which apply when the signals are deterministic with unknown parameters. Another popular signal model treats the complex amplitudes as complex Gaussian random variables. The parameters to be estimated are then imbedded in the covariance matrix of the observations rather than in their mean value. In this section, CR bounds based on a Gaussian signal model are derived.

Angle estimation is to be based on N independent snapshots of array data. The n^{th} snapshot is of the form

$$\underline{z}_n = \sum_{k=1}^{K} p_{nk}\underline{v}(u_k) + \underline{\xi}_n = V(\underline{u})\,\underline{p}_n + \underline{\xi}_n$$

Both $\underline{\xi}_n$ and $\underline{p}_n$ are complex circular Gaussian random vectors, i.e.,

$$E(\underline{\xi}_n) = E(\underline{\xi}_n \underline{\xi}_n^T) = 0,\ E(\underline{\xi}_n \underline{\xi}_n^H) = \Lambda_0$$
$$E(\underline{p}_n) = E(\underline{p}_n \underline{p}_n^T) = 0,\ E(\underline{p}_n \underline{p}_n^H) = P$$

Consequently $\underline{z}_n$ is also complex circular Gaussian.

$$E(\underline{z}_n) = E(\underline{z}_n \underline{z}_n^T) = 0,\ E(\underline{z}_n \underline{z}_n^H) = VPV^H + \Lambda_0 \triangleq \Lambda$$

The likelihood function for the observations is

$$p(\underline{z}_1, \cdots \underline{z}_n \,/\, \underline{u}, P) = \frac{1}{\pi^{MN}|\Lambda|^N} e^{-\sum_n \underline{z}_n^H \Lambda^{-1} \underline{z}_n}$$

with logarithm

$$\ell n\, p(\underline{z}_1, \cdots \underline{z}_n \,/\, \underline{u}\ , P) = -MN\, \ell n\pi - N\, \ell n|\Lambda| - \sum_n \underline{z}_n^H \Lambda^{-1} \underline{z}_n$$

Let x,y denote any two of the unknown parameters.

A typical entry in the Fisher information matrix is

$$E\,\frac{\partial \ell n p}{\partial x}\,\frac{\partial \ell n p}{\partial y}$$

The derivative is

$$\frac{\partial \ell n p}{\partial x} = -\frac{N}{|\Lambda|}\frac{\partial |\Lambda|}{\partial x} - \sum_{n=1}^{N} \underline{z}_n^H \frac{\partial \Lambda^{-1}}{\partial x} \underline{z}_n$$

Making use of the relations [4]

$$\frac{\partial |\Lambda|}{\partial x} = |\Lambda| \mathrm{Tr}\{\Lambda^{-1} \frac{\partial \Lambda}{\partial x}\}$$

$$\frac{\partial \Lambda^{-1}}{\partial x} = -\Lambda^{-1} \frac{\partial \Lambda}{\partial x} \Lambda^{-1}$$

this becomes

$$\frac{\partial \ell np}{\partial x} = \sum_{n=1}^{N} \underline{z}_n^H \frac{\partial \Lambda^{-1}}{\partial x} \underline{z}_n - N \mathrm{Tr}\{\Lambda^{-1} \frac{\partial \Lambda}{\partial x}\}$$

$$= N \mathrm{Tr}\, \Lambda^{-1} \frac{\partial \Lambda}{\partial x} \Lambda^{-1}(S-\Lambda)$$

where $S = \frac{1}{N} \sum_{n=1}^{N} \underline{z}_n \underline{z}_n^H$ is the sample covariance matrix. Since $E(S) = \Lambda$, we have

$$E\left(\frac{\partial \ell np}{\partial x}\right) = 0.$$

The desired expectation is

$$E\left(\frac{\partial \ell np}{\partial x} \frac{\partial \ell np}{\partial y}\right) = N \mathrm{Tr}\left(\Lambda^{-1} \frac{\partial \Lambda}{\partial x} \Lambda^{-1} \frac{\partial \Lambda}{\partial y}\right) \qquad (2.23)$$

This result is derived in detail in [5], Appendix 2.

Assume first that the "signal in space" covariance matrix P is completely known, and evaluate the information matrix for estimating $\underline{u}$. The required derivatives are

$$\frac{\partial \Lambda}{\partial u_i} = \frac{\partial V}{\partial u_i} P V^H + V P \frac{\partial V^H}{\partial u_i}$$

and since

$$\frac{\partial V}{\partial u_i} = (\underline{0}, \ldots, \underline{0}, \frac{\partial \underline{v}(u_i)}{\partial u_i}, \underline{0}, \ldots \underline{0}) = \dot{\underline{v}}(u_i)\, e_i^H = \dot{V} \underline{e}_i \underline{e}_i^H$$

where $\underline{e}_i$ is a unit vector whose 1^{th} element is unity, and

$$\dot{V} = (\dot{\underline{v}}(u_1), \dot{\underline{v}}(u_2), \ldots \dot{\underline{v}}(u_k))$$

we have

$$\frac{\partial \Lambda}{\partial u_i} = \dot{V}\, \underline{e}_i\, \underline{e}_i^H\, P\, V^H + VP\, \underline{e}_i\, \underline{e}_i^H\, \dot{V}^H$$

The ik^{th} element of the information matrix is given by (2.23) as

$$F_{ik} = E\left(\frac{\partial \ell np}{\partial u_i}\, \frac{\partial \ell np}{\partial u_k}\right)$$

$$= N\, Tr\{\Lambda^{-1}(\dot{V}\, \underline{e}_i\, \underline{e}_i^H\, PV^H + VP\underline{e}_i\, \underline{e}_i^H \dot{V}^H)\, \Lambda^{-1}\, (\dot{V}\, \underline{e}_k\, \underline{e}_k^H\, PV^H + VP\underline{e}_k\, \underline{e}_k^H \dot{V}^H)\}$$

As in Section C, we define the matrices

$$W = V^H \Lambda^{-1} V$$
$$\dot{W} = V^H \Lambda^{-1} \dot{V}$$
$$\ddot{W} = \dot{V}^H \Lambda^{-1} \dot{V}$$

Since any cyclic permutation of matrices does not affect the trace operation, the preceding expression can be rewritten as

$$F_{ik} = N\, Tr\{(\underline{e}_i^H \dot{P}\dot{W}\, \underline{e}_k)\, (\underline{e}_k^H \dot{P}\dot{W}\, \underline{e}_i) + \text{complex conjugate}$$
$$+(\underline{e}_i^H \ddot{W}\, \underline{e}_k)\, (\underline{e}_k^H\, PWP\underline{e}_i) + \text{complex conjugate}\}$$

The quantities in parenthesis are scalars, so the trace operator can be dropped. Also each term can be combined with its complex conjugate to give

$$F_{ik} = 2\, N\, Re\, \{(\underline{e}_i^H \dot{P}\dot{W}\, \underline{e}_k)\, (\underline{e}_k^H \dot{P}\dot{W}\, \underline{e}_i) + (\underline{e}_i^H \ddot{W}\, \underline{e}_k)\, (\underline{e}_k^H\, PWP\, \underline{e}_i)\}$$

Define the Hadamard product of two MXN matrices to be

$$(A \diamond B)_{mn} = A_{mn} B_{mn}$$

Then we can write the matrix $[F_{ik}]$ as

$$F_{\underline{u}\,\underline{u}} = 2N \text{ Re } \{P\dot{W} \diamond (P\dot{W})^T + \ddot{W} \diamond (PWP)^T\} \tag{2.24}$$

The result can be put in a more convenient form [3]. Seek an inverse for Λ of the form

$$\Lambda^{-1} = \Lambda_o^{-1} - \Lambda_o^{-1} V Q V^H \Lambda_o^{-1}$$

Computation yields

$$\Lambda\Lambda^{-1} = I + V(P-Q-PW_oQ)V^H\Lambda_o^{-1} = I$$

$$\Lambda^{-1}\Lambda = I + \Lambda_o^{-1}V(P-Q-QW_oP)V^H = I$$

where $W_o = V^H\Lambda_o^{-1}V$.

Thus Q must satisfy

$$P-Q=PW_oQ=QW_oP \tag{2.241}$$

Solving for Q, we find

$$Q = (I + PW_o)^{-1} P=P(I + W_oP)^{-1}$$

and, if P^{-1} exists

$$Q = (P^{-1} + W_o)^{-1}$$

Computation of the elements of (2.24) with frequent use of (2.241) yields

$$F = 2N \text{ Re } \{\ddot{W}_o - \dot{W}_o^H Q \dot{W}_o) \diamond (P-Q)^T + Q\dot{W}_o \diamond (Q\dot{W}_o)^T\} \qquad (2.242)$$

where

$$\dot{W}_o = V^H \Lambda_o^{-1} \dot{V}$$

$$\ddot{W}_o = \dot{V}^H \Lambda_o^{-1} \dot{V}$$

This is the desired result for the case of Gaussian signals with known covariance.

As the smallest (diagonal) element of P becomes large, $Q \to W_o^{-1}$, $P-Q \to P$ and (2.242) becomes asymptotically.

$$F = 2N \text{ Re}\{(\ddot{W}_o - \dot{W}_o^H W_o^{-1} \dot{W}_o) \diamond P\} \qquad (2.243)$$

In the case of unknown covariance P, indexing becomes a problem. We shall restrict our attention to one simple limiting case, namely, completely incoherent signals (diagonal P). We again make use of (2.23). The necessary derivatives are

$$\frac{d\Lambda}{dP_{ii}} = \frac{d}{dP_{ii}} \{\Lambda_0 + \sum_m P_{mm} \underline{v}(u_m)\underline{v}^H(u_m)\} = V\underline{e}_i \underline{e}_i^H V^H$$

We then compute

$$E\left(\frac{\partial \ell np}{\partial P_{ii}} \frac{\partial \ell np}{\partial P_{kk}}\right) = N \text{ Tr}\{\Lambda^{-1}(V \underline{e}_i \underline{e}_i^H V^H) \Lambda^{-1}(V \underline{e}_k \underline{e}_k^H V^H)\}$$

$$= N \left|\underline{e}_i^H V^H \Lambda^{-1} V \underline{e}_k\right|^2$$

or

$$E\left(\frac{\partial \ell np}{\partial \underline{P}}\right)^T \frac{\partial \ell np}{\partial \underline{P}} = N (W \diamond W^T)$$

and

$$E\{\frac{\partial \ell np}{\partial P_{ii}} \frac{\partial \ell np}{\partial u_k}\} = N \, Tr\{\Lambda^{-1} V \underline{e}_i \underline{e}_i^H V^H \Lambda^{-1} (\dot{V} \underline{e}_k \underline{e}_k^H P V^H + V P \underline{e}_k \underline{e}_k^H \dot{V}^H)\}$$

$$= 2N \, Re\{(\underline{e}_i^H V^H \Lambda^{-1} \dot{V} \underline{e}_k)(\underline{e}_k^H P V^H \Lambda^{-1} V \underline{e}_i)\}$$

or

$$E\{(\frac{\partial \ell np}{\partial \underline{P}})^T \frac{\partial \ell np}{\partial \underline{u}}\} = 2N \, Re\{\dot{W} \diamond (PW)^T\}$$

The overall information matrix for a diagonal covariance matrix P then takes the form

$$F = \begin{bmatrix} F_{\underline{P}\,\underline{P}} & F_{\underline{P}\,\underline{u}} \\ F_{\underline{u}\,\underline{P}} & F_{\underline{u}\,\underline{u}} \end{bmatrix} = N \begin{bmatrix} W \diamond W^T & 2 \, Re\{\dot{W} \diamond (PW)^T\} \\ \text{symmetric} & 2 \, Re\{\ddot{W} \diamond (PWP)^T + \dot{P}\dot{W} \diamond (P\dot{W})^T\} \end{bmatrix} \quad (2.25)$$

Use of (2.241) gives the more convenient form

$$F = N \begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} QW_o \diamond (QW_o)^T & 2 \, Re\{Q\dot{W}_o \diamond (QW_o)^T\} \\ \text{symmetric} & 2 \, Re\{(\ddot{W}_o - \dot{W}_o^H Q \dot{W}_o) \diamond (P-Q)^T + Q\dot{W}_o \diamond (Q\dot{W}_o)^T\} \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \quad (2.251)$$

provided that P^{-1} exists.

G. Examples of CR Bounds for Gaussian Signal Model

One Emitter in White Noise

The covariance matrix for this case is

$$\Lambda = \sigma^2 (I + \rho \, \underline{v} \, \underline{v}^H)$$

where ρ is the array signal-to-noise ratio per observation. The inverse is

$$\Lambda^{-1} = \frac{1}{\sigma^2} (I - \frac{\rho}{\rho+1} \, \underline{v} \, \underline{v}^H)$$

We compute

$$W = \underline{v}^H \Lambda^{-1} \underline{v} = \frac{1}{\sigma^2} (1 - \frac{\rho}{\rho+1}) = \frac{1}{\sigma^2(\rho+1)}$$

$$\dot{W} = \underline{v}^H \Lambda^{-1} \dot{\underline{v}} = \frac{1}{\sigma^2} (1 - \frac{\rho}{\rho+1}) \, \underline{v}^H \dot{\underline{v}} = \frac{1}{\sigma^2(\rho+1)} \frac{j}{M} \sum \frac{x_m}{\lambda}$$

$$\ddot{W} = \dot{\underline{v}}^H \Lambda^{-1} \dot{\underline{v}} = \frac{1}{\sigma^2} \dot{\underline{v}}^H \dot{\underline{v}} - \frac{\rho}{\sigma^2(\rho+1)} |\underline{v}^H \dot{\underline{v}}|^2$$

We assume as before that the array coordinate system is chosen so that $\sum x_m = 0$.

Then

$$\dot{W} = 0, \; \ddot{W} = \frac{1}{M\sigma^2} \sum (\frac{2\pi x_m}{\lambda})^2 = (\frac{2\pi \ell}{\sigma\lambda})^2$$

Equation (2.24) becomes

$$F_{uu} = 2N \frac{(\sigma^2\rho)^2}{\sigma^2(\rho+1)} \frac{(2\pi\ell)^2}{\sigma^2} = \frac{2N\rho}{1+1/\rho} (\frac{2\pi\ell}{\lambda})^2$$

Therefore the CR bound for estimating the array coordinate u of one Gaussian signal form N independent observations is

$$\sigma_u^2 \geq \frac{1+1/\rho}{2N\rho(2\pi\ell)^2} \tag{2.27}$$

which is the same as the bound for a deterministic signal (2.17) except for

the factor $1+1/\rho$. Note that the bound is the same whether the signal power is known or unknown.

One Emitter in Noise and Known Interference

This differs from a previous example in that the desired signal is modelled as Gaussian rather than deterministic.

The covariance matrix is

$$A = R + P\,\underline{v}\,\underline{v}^H$$

where R is the known covariance matrix of the interference and noise and P is the unknown power of the emitter. For brevity, define

$$r = \underline{v}^H R^{-1}\,\underline{v}$$

$$\dot{r} = \underline{v}^H R^{-1}\,\dot{\underline{v}}$$

$$\ddot{r} = \dot{\underline{v}}^H R^{-1}\,\dot{\underline{v}}$$

Then

$$A^{-1} = R^{-1} - \frac{P}{1+Pr} R^{-1}\,\underline{v}\underline{v}^H R^{-1}$$

The required matrices (scalars) are

$$W = \frac{r}{1 + Pr}$$

$$\dot{W} = \frac{\dot{r}}{1 + Pr}$$

$$\ddot{W} = \ddot{r} - \frac{P}{1 + Pr}|\dot{r}|^2$$

Using (2.25) we obtain the information matrix for the parameters P, u

respectively.

$$F = \frac{N}{(1+Pr)^2}\begin{bmatrix} r^2 & 2Pr\ Re(\dot{r}) \\ \text{symmetric} & 2P^2[(1+Pr)(\ddot{r}\bar{r}-|\dot{r}|^2) + 2\ Re^2(\dot{r})] \end{bmatrix} \quad (2.28)$$

This agrees with a result obtained by Miller and Huber [6].

The reduced information matrix for u alone is

$$F_u^{(-)} = 2\frac{P^2}{1+Pr}(\ddot{r}\bar{r} - |\dot{r}|^2) \quad (2.29)$$

To compare this with a previous result (2.22) which was derived using a deterministic signal model, we take

$$R = \sigma^2 (I + \rho_I \underline{v}_I \underline{v}_I^H)$$

The interference then consists of white noise plus a single source of Gaussian interference at known direction u_I. Then

$$R^{-1} = \frac{1}{\sigma^2}(I - \gamma \underline{v}_I \underline{v}_I^H)$$

where $\gamma = \rho_I/(\rho_I+1)$.

We evaluate

$$r = \underline{v}_s^H R^{-1} \underline{v}_s = \frac{1}{\sigma^2}(1-\gamma|\underline{v}_I^H \underline{v}_s|^2)$$

$$\dot{r} = \underline{v}_s^H R^{-1} \dot{\underline{v}}_s = -\frac{\gamma}{\sigma^2}(\underline{v}_s^H \underline{v}_I)(\underline{v}_I^H \dot{\underline{v}}_s)$$

$$\ddot{r} = \frac{(2\pi\ell)^2}{\sigma^2} - \frac{\gamma}{\sigma^2}|\underline{v}_I^H \ddot{\underline{v}}_s|^2$$

Substitution of these results into (2.29) gives

$$P_u^{(-)} = \frac{2\rho_s\left[(2\pi\ell)^2 - \dfrac{\gamma|\underline{v}_I^H\underline{v}_s|^2}{1-\gamma|\underline{v}_s^H\underline{v}_I|^2}\right]}{1 + \dfrac{1}{\rho_s(1-\gamma|\underline{v}_I^H\underline{v}_s|^2}} \qquad (2.30)$$

where $\rho_s = \frac{P}{\sigma^2}$.

The reciprocal of (2.30) agrees with (2.22) except for a factor (the denominator of (2.30)) which approaches unity for large signal-to-noise ratio.

2. ZIV-ZAKAI BOUNDS [7]

A. One Unknown Parameter

Let u be an unknown parameter taking values in an interval U. The value of u is to be estimated based on some set of observations which may be written as a vector $\underline{z}$. A statistical model is given relating the observations to the value of u.

For any two values of u, u_1, and $u_2 > u_1$, there exists an optimal statistical decision rule, the likelihood ratio test (LRT), for deciding on the basis of the observations which of the two values is correct. The LRT is optimal in the sense that it minimizes the probability of error.

If the values u_1, u_2 are equally likely to occur a priori*, the LRT chooses u_1 if

$$\ell \triangleq \ell n \frac{p(\underline{z}|u_1)}{p(\underline{z}|u_2)} > 0$$

and otherwise u_2. The probability of error for this optimal test is

$$P_E = \frac{1}{2} \Pr (\ell < 0 \mid u_1) + \frac{1}{2} \Pr (\ell > 0 \mid u_2)$$

Note that P_E depends on the two values u_1, u_2, and also on the statistics of the observations.

Now consider a suboptimal decision rule based on an arbitrary estimate $\hat{u}$ of u. This rule chooses u_1 if $\hat{u}$ is closer to u_1 than to u_2, and vice versa. The error probability of this decision rule is

---

*It is possible to derive a more general form of the ZZ bound by allowing arbitrary a priori probabilities [8]. However, this tends to obscure the basic ideas involved and furthermore provides no improvement in the bound for the examples to be discussed here. See Appendices II, III.

$$Q_E = \frac{1}{2}\Pr\left(\hat{u} > \frac{u_1+u_2}{2} \mid u_1\right) + \frac{1}{2}\Pr\left(\hat{u} < \frac{u_1+u_2}{2} \mid u_2\right)$$
$$< \frac{1}{2}\Pr\left(|\hat{u}-u_1| > \frac{|u_2-u_1|}{2} \mid u_1\right) + \frac{1}{2}\Pr\left(|\hat{u}-u_2| > \frac{|u_2-u_1|}{2} \mid u_2\right)$$

Application of the Chebychev inequality yields

$$Q_E < \frac{1}{2}\,\frac{e^2(u_1)}{\left(\frac{u_2-u_1}{2}\right)^2} + \frac{1}{2}\,\frac{e^2(u_2)}{\left(\frac{u_2-u_1}{2}\right)^2}$$

where $e^2(u)$ denotes the mean squared error in the estimate $\hat{u}$ when u is the true value. Since $Q_E > P_E$, we have

$$\frac{1}{2}\left(e^2(u_1) + e^2(u_2)\right) > \left(\frac{u_2-u_1}{2}\right)^2 P_E(u_1, u_2) \tag{2.31}$$

for any pair of values u_1, u_2.

The right-hand side of (2.31) has been increased by a factor of 2 by Wax and Ziv [9] using a much more complicated argument (see Appendix I). We incorporate this improvement in the remainder of this report. Thus, the basic Ziv-Zakai bound is

$$\frac{1}{2}\left(e^2(u_1) + e^2(u_2)\right) > \frac{(u_2-u_1)^2}{2} P_E(u_1, u_2) \tag{2.32}$$

Observe that the ZZ bound is a lower bound on the average mean squared estimation error at <u>two</u> parameter values. This is the best one can hope for when considering a completely unrestricted class of estimators, since the estimate $\hat{u} = u_0$ has zero error when $u = u_0$. To obtain a point bound, note that for any given estimator $\hat{u}$ there exists some value u_* which produces the largest mean square error. The left hand side of (2.32) is then less than $e^2(u_*)$ for any choice of u_1, u_2 and so we can maximize the right-hand side over these variables, getting

$$e^2(u_*) > \max_{u_1, u_2} \frac{(u_1 - u_2)^2}{2} P_E(u_1, u_2) \qquad (2.33)$$

which is a lower bound on the worst possible error any estimator can make.

B. Additional Unknown Parameters

The argument leading up to the two-point bound (2.32) is unchanged by the presence of other unknown parameters in the statistical model*. Let $\underline{\alpha}$ denote the vector of additional parameters. The LRT chooses between two values $(u_1, \underline{\alpha}_1)$, $(u_2, \underline{\alpha}_2)$. Its error probability is a function of these values.

As before, define a suboptimal hypothesis test based on an arbitrary estimate $\hat{u}$ of u. This leads directly to a multi-parameter version of (2.32).

$$\frac{1}{2}\left[e^2_{\hat{u}}(u_1, \underline{\alpha}_1) + e^2_{\hat{u}}(u_2, \underline{\alpha}_2)\right] > \frac{(u_2 - u_1)^2}{2} P_E(u_1, u_2, \underline{\alpha}_1, \underline{\alpha}_2) \qquad (2.34)$$

There are many ways to obtain point bounds from (2.34). The parameter vector $\underline{\alpha}$ can be divided into a vector $\underline{\beta}$ of "interesting" parameters and a vector $\underline{\gamma}$ of "nuisance" parameters $\underline{\gamma}$. The distinction is that we wish to obtain a bound which is an explicit function of the "interesting" parameters, but are willing to choose worst-case values $\underline{\gamma}_*(\underline{\beta})$ for the "nuisance" parameters (and u). This gives

$$\frac{1}{2}\left[e^2_{\hat{u}}(u_*(\underline{\beta}_1), \underline{\beta}_1, \underline{\gamma}_*(\underline{\beta}_1)) + e^2_{\hat{u}}(u_*(\underline{\beta}_2), \underline{\beta}_2, \underline{\gamma}_*(\underline{\beta}_2))\right]$$
$$> \frac{(u_2 - u_1)^2}{2} P_E(u_1, u_2, \underline{\beta}_1, \underline{\beta}_2, \underline{\gamma}_1, \underline{\gamma}_2)$$

---

*Known parameters are simply part of the statistical model for the observations and need not be considered explicitly.

The inequality is valid for any choice of $\underline{\beta}_1$, $\underline{\beta}_2$; however, it is most useful when we choose $\underline{\beta}_1 = \underline{\beta}_2 = \underline{\beta}$ to get a point bound

$$e^2_{\underset{u}{\wedge}}\left(u_*(\underline{\beta}),\ \underline{\beta},\ \underline{\gamma}_*(\underline{\beta})\right) \geq \frac{(u_2 - u_1)^2}{2} P_E(u_1,\ u_2,\ \underline{\beta},\ \underline{\gamma}_1,\ \underline{\gamma}_2)$$

This inequality holds for any choice of u_1, u_2, $\underline{\gamma}_1$, $\underline{\gamma}_2$; as in the one-parameter case, the tightest bound is obtained by maximizing the right-hand side with respect to these variables. The result is

$$e^2_{\underset{u}{\wedge}}\left(u_*(\underline{\beta}),\ \underline{\beta},\ \underline{\gamma}_*(\underline{\beta})\right) \geq \max_{u_1,u_2} \frac{(u_2 - u_1)^2}{2} \max_{\underline{\gamma}_1,\underline{\gamma}_2} P_E(u_1,\ u_2,\ \underline{\beta},\ \underline{\gamma}_1,\ \underline{\gamma}_2) \tag{2.35}$$

The problems to be considered in this report are signal parameter estimation problems characterized by the observation model

$$\underline{z} = \sum_{i=1}^{K} A_i e^{j\phi_i}\, \underline{v}(u_i) + \underline{\xi}$$

To get a meaningful bound on the error in estimating u_i, the associated signal amplitude A_i must be treated as an "interesting" parameter. The reason is that the worst-case value of A_i is clearly zero. The signal is then absent and the observation provides no information about u_i. The other amplitudes, however, may be treated as "nuisance" parameters.

The Cramer-Rao bound clearly shows how the presence of additional unknown parameters degrades the bound on accuracy for estimating a parameter. With the Ziv-Zakai bound, the distinction is less clear. The parameters $\underline{\beta}$ in (2.35) are held fixed in the two hypotheses being tested; they are thus treated in the same way as the known parameters. The parameters $\underline{\gamma}$ are different in the two hypotheses being tested and therefore must be unknown

parameters. The bound (the right-hand side of (2.35)) is an explicit function of parameters $\underline{\beta}$ and makes no distinction between whether they are known or unknown; it is an implicit function of unknown parameters $\underline{\gamma}$ for which worst-case values have been selected.

We now specialize the bound (2.35) to some cases of particular interest. Paralleling the development of Cramer-Rao bounds, we begin with deterministic signal models and then treat the Gaussian signal model.

C. One Known Signal in White Gaussian Noise

The observation is a complex Gaussian random vector with covariance matrix $\sigma^2 I$ under either hypothesis; its mean is

$$Ae^{j\phi}\underline{s}(u) \quad .$$

The error probability for deciding between two values u_1 and u_2 when A and ϕ are known is (see Appendix II)[1].

$$P_E = \text{erfc}_* \left[\frac{A}{\sqrt{2}\,\sigma} \left| \underline{s}(u_1) - \underline{s}(u_2) \right| \right]$$

Of particular interest is the case where the length of the signal vector $\underline{s}$ is unity for any parameter value, and the projection $\underline{s}(u_1)^H\underline{s}(u_2)$ depends only on the parameter difference $u_1 - u_2$. These assumptions hold, for example, in the time-of-arrival and direction-finding problems when only one signal is present.

Then the error probability becomes

$$P_E = \text{erfc}_* \left[\frac{A}{\sigma} \sqrt{1 - \text{Re}[c(u_1 - u_2)]} \right]$$

---

[1] $\text{erfc}_*(x) \triangleq \frac{1}{\sqrt{2\pi}\,\sigma} \int_x^\infty e^{-\frac{r^2}{2}}\, dr$

where $|\underline{s}(u)|^2 \triangleq 1$ and $c(u_1 - u_2) \triangleq \underline{s}(u_1)^H \underline{s}(u_2)$. The function c is a measure of the correlation, or degree of similarity, between normalized signals having different parameter values. The ZZ bound for this case is then

$$\frac{1}{2}\left[e^2(u_1,A) + e^2(u_2,A)\right] \geq \frac{(u_1 - u_2)^2}{2} \operatorname{erfc}_*\left[\frac{A}{\sigma}\sqrt{1 - \operatorname{Re}\{c(u_1 - u_2)\}}\right] \tag{2.36}$$

and the worst-case Z bound is

$$e^2(u_*,A) \geq \max_{u_1} \max_{u_2} \frac{(u_1 - u_2)^2}{2} \operatorname{erfc}_*\left(\frac{A}{\sigma}\sqrt{1 - \operatorname{Re}\{c(u_1 - u_2)\}}\right) \tag{2.37}$$

The right-hand side of (2.37) involves two competing terms - the first grows quadratically, while the second usually becomes small as the parameter values separate. In the time-of-arrival problem, for example, $c(u_1 - u_2)$ is the normalized correlation function of the pulse envelope, which typically decreases smoothly and becomes zero once the time difference $u_1 - u_2$ exceeds the pulse width. Typical behavior of the right-hand side of (2.37) is shown as a function of parameter separation for increasing values of signal-to-noise ratio in Fig. 2.1. The non-monotonic behavior reflects that of the signal correlation function c.

At large signal-to-noise ratios, the maximum required on the right-hand side of (2.37) occurs when $u_1 - u_2$ is small, as shown in Fig. 2.2. The argument of the error function can be bounded for small $\varepsilon = u_1 - u_2$ by using the inequality $\cos x \geq 1 - 0.5\, x^2$.

$$\frac{A}{\sigma}\sqrt{1 - \operatorname{Re}\left\{\frac{1}{M}\sum e^{j2\pi\frac{x_m}{\lambda}\varepsilon}\right\}} = \frac{A}{\sigma}\sqrt{1 - \frac{1}{M}\sum \cos 2\pi\varepsilon\frac{x_m}{\lambda}} \leq \frac{A}{\sigma}\sqrt{1 - \frac{1}{M}\sum \left(1 - \frac{1}{2}\left(\frac{2\pi x_m \varepsilon}{\lambda}\right)^2\right)}$$

$$= \frac{A}{\sqrt{2}\,\sigma}\, 2\pi\, \ell\, \varepsilon \triangleq x$$

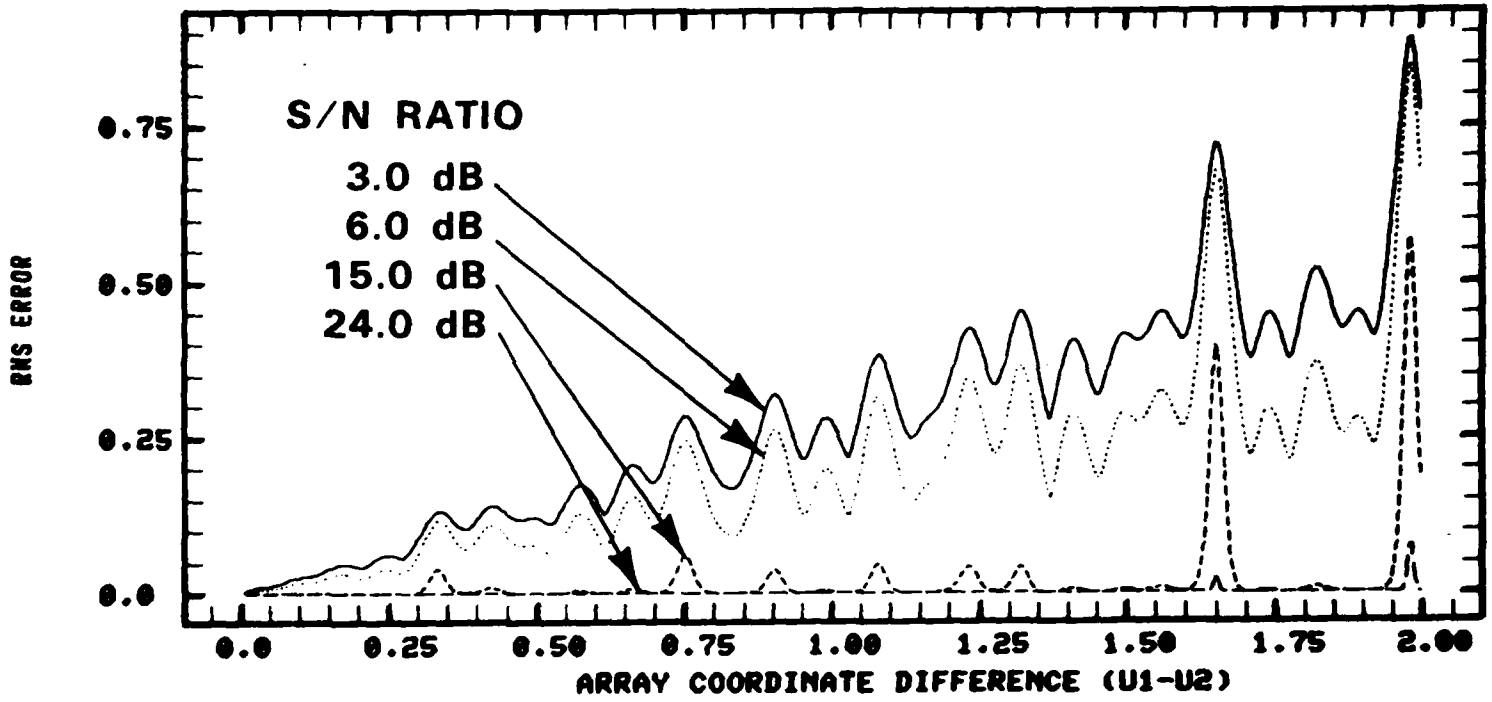


Fig. 2.1. Behavior of Ziv-Zakai bound vs. emitter separation for increasing S/N ratio.

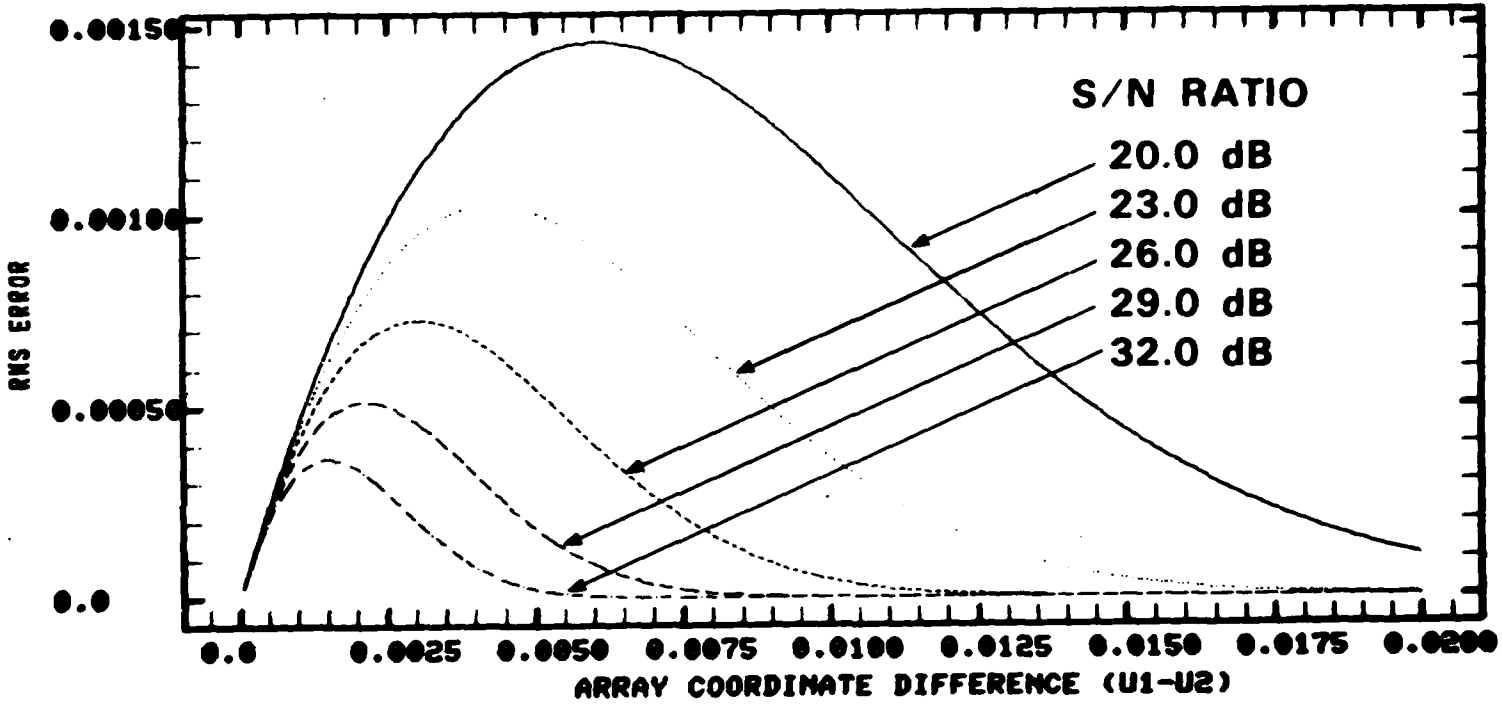


Fig. 2.2. Peak value of Ziv-Zakai bound for large S/N ratio.

where

$$\ell^2 = \frac{1}{M} \sum_{m=1}^{M} \left(\frac{x_m}{\lambda}\right)^2$$

is the same mean squared array length which appears in the Cramer-Rao bound (see (2.17)).

The right-hand side of (2.37) is then greater than

$$\left(\frac{\sigma}{2A\pi\ell}\right)^2 x^2 \mathrm{erfc}_* x$$

which has a maximum over x of 0.16572 at x = 0.84188.

Thus, for large S/N, the worst-case ZZ bound becomes

$$e^2(u_*) \geq 0.16572 \frac{\sigma^2}{A^2} \frac{1}{(2\pi\ell)^2} \qquad (2.38)$$

This bound is smaller (weaker) by a factor of 0.33044 (4.8 dB) than the Cramer-Rao bound.

D. One Signal with Unknown Phase in White Gaussian Noise

The contending hypotheses are

$$H_1: \quad \underline{z} = Ae^{j\phi_1}\underline{s}(u_1) + \underline{\xi}$$

$$H_2: \quad \underline{z} = Ae^{j\phi_2}\underline{s}(u_2) + \underline{\xi}$$

where, $E(\underline{\xi}) = E(\underline{\xi}\,\underline{\xi}^T) = 0$, $E(\underline{\xi}\,\underline{\xi}^H) = \sigma^2 I$.

The derivation of the minimum error probability for deciding between H_1 and H_2 is given in Appendix III. Assuming $|\underline{s}(u_1)|^2 = 1$, $\underline{s}^H(u_1)\ \underline{s}(u_2) = c(u_1 - u_2)$ as in the preceding section, the result is

$$P_E(u_1, u_2) = \text{erfc}_*\left(\frac{A}{\sigma}\sqrt{1 - |c(u_1 - u_2)|}\right)$$

and the corresponding worst-case bound is

$$e^2(u_*) \geq \max_{u_1,u_2} \frac{(u_1 - u_2)^2}{2}\ \text{erfc}_*\left(\frac{A}{\sigma}\sqrt{1 - |c(u_1 - u_2)|}\right) \qquad (2.39)$$

Thus, the only change to the bound (2.37) is that Re c is replaced by $|c|$.

Note that by treating phase as an unknown (nuisance) parameter, we obtain a bound which differs from that for known phase and is independent of the actual phase value.

It was pointed out previously that at large S/N ratios, the desired maximum occurs for small $u_1 - u_2$; this implies $|c(u_1 - u_2)| \approx \text{Re}\ c(u_1 - u_2) \approx 1$. Thus, the worst-case ZZ bounds for known and unknown phase coincide for large S/N ratios.

E. One Signal with Random Phase in White Gaussian Noise

A third signal model of interest is one that treats phase as a random variable uniformly distributed as $(0, 2\pi)$. It can be shown [8] that the optimum hypothesis test between equally likely parameter values u_1 and u_2 has error probability

$$P_E = S(\alpha,\beta) = \frac{1}{2}\left[1 - Q\left(\sqrt{\beta}, \sqrt{\alpha}\right)\right] + \frac{1}{2} Q\left(\sqrt{\alpha}, \sqrt{\beta}\right)$$

where

$$\alpha = \frac{A^2}{2\sigma^2}\left(1 - \sqrt{1 - \left|c(u_1 - u_2)\right|^2}\right)$$

$$\beta = \frac{A^2}{2\sigma^2}\left(1 + \sqrt{1 - \left|c(u_1 - u_2)\right|^2}\right)$$

and

$$Q(a, b) = \int_b^\infty x e^{-\frac{x^2 + a^2}{2}} I_o(ax)\, dx$$

is Marcum's Q function. The resulting worst-case ZZ bound is

$$e^2(u_*) \geq \max_{u_1 - u_2} \frac{(u_1 - u_2)^2}{2} S(\alpha, \beta) \qquad (2.40)$$

For large S/N ratios, the expression for the error probability is asymptotically equal to erfc* $\left(\sqrt{\beta} - \sqrt{\alpha}\right)$ and the right-hand side of (2.40) becomes (approximately)

$$\max_u \frac{u^2}{2} \operatorname{erfc}_*\left(\frac{A}{\sqrt{2}\,\sigma}\sqrt{1 + \sqrt{1 - \left|c(u)\right|^2}} - \frac{A}{\sqrt{2}\,\sigma}\sqrt{1 - \sqrt{1 - \left|c(u)\right|^2}}\right)$$

At high S/N ratios, the maximum occurs for small u, so that $|c(u)| \approx 1$, and the above expression is approximately

$$\max_u \frac{u^2}{2} \operatorname{erfc}_*\left(\frac{A}{\sigma}\sqrt{1 - |c(u)|}\right)$$

Thus, at high S/N ratio, the random phase bound agrees with the unknown phase bound.

F. Multiple Elementary Signals with Unknown Phases in White Gaussian Noise

The observation is modelled as

$$\underline{z} = V(\underline{u})\ \underline{p} + \underline{\xi}$$

The columns of V are the elementary signals $\underline{v}(u_i)$ and $\underline{p}$ is a vector of unknown complex amplitudes. The minimum error probability for deciding between $(\underline{u},\underline{p})$ and $[\underline{u}', \underline{p}']$ in this case is derived in Appendix IV. The result is

$$P_E = \operatorname{erfc}_*\left(\frac{Ad}{\sqrt{2}}\right)$$

where

$$d = f(\underline{u}, \underline{u}', \underline{p}, \underline{p}')$$

The resulting worst-case ZZ bound on the error in estimating the angle-of-arrival u_n of the nth signal is

$$e^2(u_{n*}, A_n) \geq \max_{\underline{u},\underline{u}',\underline{p},\underline{p}'} \frac{(u_n - u_n')^2}{2} \operatorname{erfc}_*\left(\frac{A_n d}{\sqrt{2}\,\sigma}\right) \qquad (2.41)$$

The maximization over p, p' can be done analytically, as discussed in Appendix IV. However, the maximization over u, u' does not appear tractable for more than one signal.

G. One Gaussian Signal in White Gaussian Noise

The contending hypotheses are

$$H_1 : \underline{z}_n = p_n \underline{v}(u_1) + \underline{\xi}_n$$
$$H_2 : \underline{z}_n = p_n \underline{v}(u_2) + \underline{\xi}_n \qquad n = 1, \ldots N$$

where $\underline{\xi}_n$ is a complex Gaussian noise vector with zero mean and covariance matrix $\sigma^2 I$, and p_n is a complex Gaussian random variable with zero mean and variance MP. Both $\underline{\xi}_n$ and p_n are independent from look to look.

The minimum error probability for deciding whether H_1 or H_2 is true is shown in Appendix V to be

$$P_E = \Pr \{U > V\}$$

where U, V are independent chi-squared random variables with 2N degrees of freedom and variances (per degree of freedom)

$$\sigma_v^2 = \rho \left(1 + 2\mathrm{Re}(A^* \underline{v}_1^H \underline{v}_2) + \frac{1 + \rho |\underline{v}_1^H \underline{v}_2|^2}{1 + \rho} |a|^2\right)$$

$$\sigma_u^2 = \rho \left(|a|^2 + 2\mathrm{Re}(A^* \underline{v}_1^H \underline{v}_2) + \frac{1 + \rho\ |\underline{v}_1^H \underline{v}_2|^2}{1 + \rho}\right)$$

and

$$\rho = \frac{MP}{\sigma^2} = \text{array signal-to-noise ratio/look}$$

$$a = -\frac{1}{\lambda^*} \left(1 - \sqrt{1 - |\lambda|^2}\right)$$

$$\lambda = \frac{2\, \underline{v}_1^H \underline{v}_2}{1 + \dfrac{1 + \rho |\underline{v}_1^H \underline{v}_2|^2}{1 + \rho}}$$

It is further shown in Appendix VI that this probability is equal to the probability of obtaining less than N successes in 2N-1 trials with probability

of success

$$P_s = \frac{\sigma_v^2}{\sigma_u^2 + \sigma_v^2}$$

The resulting worst case ZZ bound is

$$e^2(u) > \max_{u_1-u_2} \frac{(u_1-u_2)^2}{2} P_E$$

It is particularly interesting to look at the result for a single snapshot (N=1). Then

$$P_E = \frac{1}{1 + \frac{\sigma_v^2}{\sigma_u^2}}$$

A rather tedious calculation shows that for large signal-to-noise ratio ρ

$$P_E \tilde{=} [(1 - |c(u)|^2)\,\rho]^{-1}$$

and the ZZ bound becomes

$$e^2(u) > \max_u \frac{1}{\rho} \frac{u^2}{1 - |c(u)|^2}$$

This is quite different from the corresponding results for deterministic signal models. In those cases, the maximizing value of u exhibits discoutinuities as signal-to-noise ratio is increased, and ultimately approaches zero. In this case, the maximizing value of u is independent of signal-to-noise ratio (provided S/N is large enough) and is not necessarily small. Thus the ZZ bound does <u>not</u> approach the CR bound at high signal-to-noise.

As the number of looks increases, the behavior of the Gaussian signal ZZ

bound approaches that of its deterministic counterpart. Fig. 2.3 shows the two sets of bounds for a uniform, 9-element linear array with $\lambda/2$ spacing, for 1, 5, 10, and 100 looks. The two bounds ultimately agree for 2 or more looks; with 20 looks they are almost identical for all signal-to-noise ratios.

The intuitive explanation of this result is that on a single trial the probability of drawing a low signal-to-noise ratio from a Gaussian sample is too high. With multiple independent observations, this situation is rectified.

The conclusion is that when multiple independent observations are available, the choice of a signal model is not critical. However, if only one observation is available, the two signal models lead to quite different results.

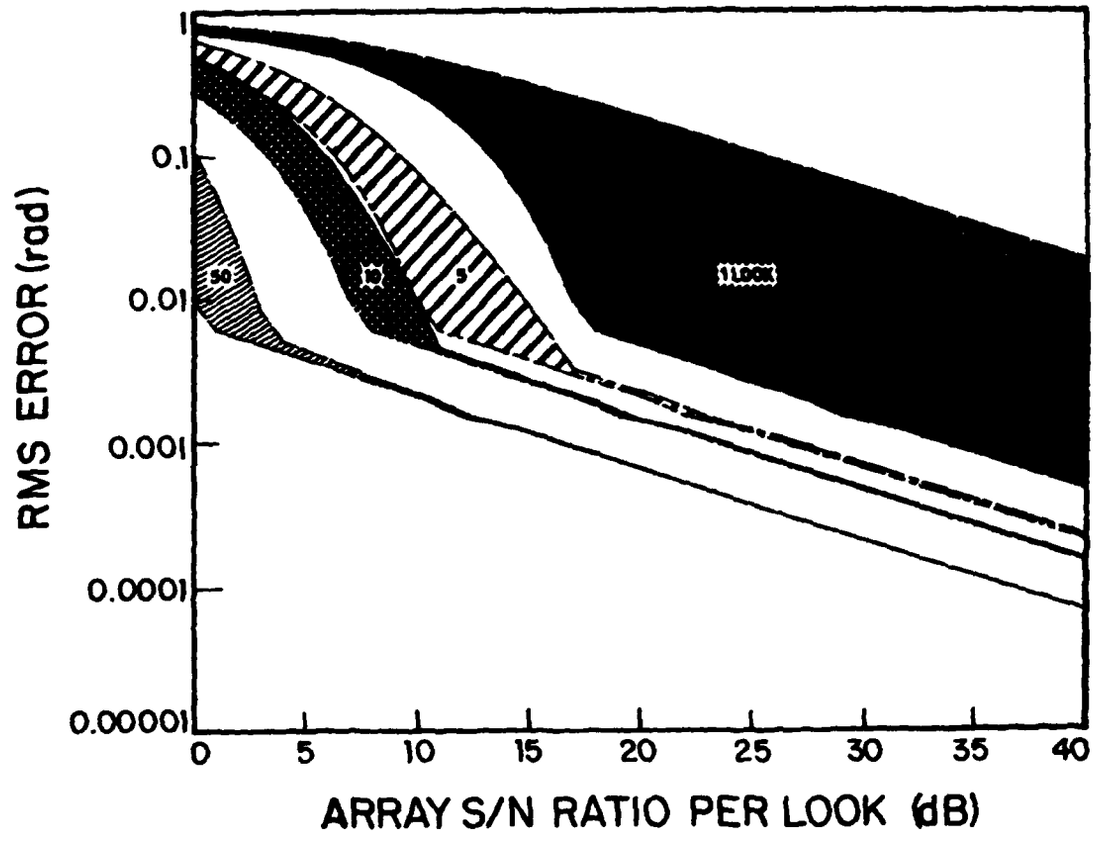


Fig. 2.3. Single emitter Ziv-Zakai bounds for Gaussian and deterministic signal models.

III. UNEQUALLY SPACED LINEAR ARRAYS

A. Applicable Methods

Many angle estimation techniques can be interpreted in terms of adaptive array concepts. The basic idea is to form a weighted sum of the array element outputs so that the total output power is minimized. Some sort of constraint must be imposed on the weighting vector to avoid the trivial solution $\underline{w} = 0$.

If the constraint is a linear one of the form $\underline{w}^H\underline{c} = 1$, then it is easily shown that the minimizing weighting vector is

$$\underline{w} = \sigma^2 R^{-1}\underline{c}$$

and the adapted array pattern in the direction $u = \sin\theta$ is

$$G(u) = |\underline{w}^H\underline{v}(u)| = \sigma^2 \, |\underline{c}^H R^{-1}\underline{v}(u)|$$

When the signal-to-noise ratio (SNR) is large enough, it can be shown that under certain conditions the adapted pattern has (asymptotic) nulls in the directions of the external sources. The required conditions are:

(1) The direction vectors of the sources are linearly independent.

(2) The signal-in-space covariance matrix P has an inverse.

These nulls can be used to determine the source directions.

If $\underline{c}^T = (1, 0, \ldots, 0)$, the constraint fixes the weight applied to one element (the "reference" element). For a uniformly spaced linear array, this method is mathematically identical to a special case of autoregressive/maximum-entropy spectral estimation in which the model order is equal to the number of array elements. While it continues to have meaning in an adaptive array context for unequally spaced arrays, it does not provide a spectral estimate having maximum entropy. Since the weights constitute the best linear prediction of the signal at the reference element based on the signals observed at all the other elements, we will refer to this method as

the linear prediction (LP) technique. In adaptive array terminology, when the reference element is the main antenna and the other samples are from auxiliary antennas, the technique is called sidelobe cancellation.

When $\underline{c} = \underline{v}(u_0)$, the constraint fixes the adapted array gain $|\underline{w}^H\underline{v}(u)|$ in the direction u_0. This pattern has (asymptotic) nulls in all source directions (except u_0), which could be used to locate them. However, the rationale behind the maximum likelihood method (MLM) is somewhat different.

Assume that the process on the aperture consists of a desired signal from direction u_0 and interference. Assume further that the signal and interference are uncorrelated. The expected power output of the array using weighting vector $\underline{w}$ is

$$E\ |\underline{w}^H\underline{z}|^2 = \underline{w}^H R\ \underline{w}$$

where $R = R_I + P_s\ \underline{v}(u_o)\ \underline{v}^H(u_o)$ is the covariance matrix of the process. The constraint $\underline{w}^H\underline{v}(u_o) = 1$ fixes the output power due to the desired signal. Minimizing the interference power output is thus equivalent to minimizing the total power output. The minimizing weighting vector is

$$\underline{w}(u_o) = \frac{R^{-1}\underline{v}(u_0)}{\underline{v}^H(u_o)\ R^{-1}\underline{v}(u_o)}$$

and the expected output power is

$$P(u_o) = \frac{1}{\underline{v}^H(u_o)R^{-1}\underline{v}(u_o)} \qquad (3.1)$$

This quantity is computed as a function of u_o and the locations of its peaks (exceeding a threshold) are the estimates of the source directions.

A third choice for the weight constraint is $\underline{w}^H\underline{w} = 1$. This fixes the thermal noise output power of the array. The minimizing weighting vector in this case satisfies the equation

$$R\underline{w} = \lambda\underline{w}$$

This eigenvalue problem has M solutions, M being the number of array elements and consequently the dimension of R. The M solution vectors are mutually orthogonal and their associated eigenvalues λ_m are real. The desired solution is the eigenvector corresponding to the minimum eigenvalue. In general, this solution is not unique. It can be shown [10] that if I < M emitters are present, R has I "signal" eigenvectors with relatively large eigenvalues and M-I "noise" eigenvectors with a common minimum eigenvalue. Any one of these noise eigenvectors is a valid solution to the minimization problem.

If the signal-in-space covariance matrix P is non-singular, it can be shown that the signal vectors $\underline{v}(u_i)$, i = 1, . . . I and the "signal" eigenvectors span the same I-dimensional subspace, which is orthogonal to the (M-I)-dimensional subspace spanned by the "noise" eigenvectors $\underline{w}_m$. It follows that the adapted array patterns $\underline{w}_m^H\underline{v}(u)$ all have nulls in each of the signal directions. When the covariance matrix is perfectly known, these are true, rather than asymptotic, nulls.

The array pattern generated by a particular "noise" eigenvector will, in general, have extraneous nulls. However, the only nulls common to <u>all</u> of the "noise" eigenvector patterns are those corresponding to source directions. This is true because if $\underline{v}(u)$ is orthogonal to <u>all</u> the "noise" eigenvectors, it must lie in the signal subspace, which is spanned by the source direction vectors $\underline{v}(u_1)$, . . ., $\underline{v}(u_I)$. But $\underline{v}(u)$ is linearly independent of these vectors unless u is one of the source directions*. The appropriate function to examine for nulls is thus

$$\sum_{m=1}^{M-I} |\underline{w}_m^H\underline{v}(u)|^2$$

*If the direction vectors are not linearly independent, extraneous nulls will exist.

for which all nulls correspond to source directions. This technique is called MUSIC (multiple signal classification) [10].

As the signal-to-noise ratio* becomes large, it was shown in [3] that

$$R^{-1} \rightarrow E_N E_N^H$$

where

$$E_N = [\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_{M-I}]$$

is a matrix whose columns are noise eigenvectors. The MLM method seeks the maxima of (3.1), or equivalently the minima of

$$\underline{v}^H(u)\, R^{-1}\, \underline{v}(\underline{u}) \rightarrow \left|E_N^H\, \underline{v}(u)\right|^2 = \sum_{m=1}^{M-I} \left|\underline{w}_\ell^H\, \underline{v}(u)\right|^2 \quad .$$

Thus, MLM is asymptotically (high S/N) equivalent to MUSIC and has no extraneous nulls. At lower S/N, however, extraneous nulls may be encountered. This is also true for the linear prediction method.

To reject whatever extraneous nulls exist, an estimate of the power received from each candidate source direction is made. Those that fall below a threshold are eliminated. The power estimate is obtained by solving the matrix equation

$$R = V\, P\, V^H + \sigma^2\, I$$

for the signal-in-space covariance matrix P, with the result

$$\hat{P} = V^+\, (R - \sigma^2\, I)\, V^{+H}$$

---

*Defined as the ratio of the smallest non-zero eigenvalue of P to the noise power σ^2.

where $V^+ = (V^H V)^{-1} V^H$ is the pseudo-inverse of the matrix † of (candidate) direction vectors V. An estimate of the noise power level σ^2 is required.

The rational for the adaptive methods collapses when the signal sources are coherent, i.e., when the signal-in-space covariance matrix P is singular. The adapted array pattern no longer has nulls in the directions of all sources. The reason is that the component of array output power due to a combination of coherent sources can be reduced to zero by cancellation, i.e., by combining them with appropriate complex weights. These weights are the values of the adapted pattern at the directions of the coherent sources.

A technique known as "spatial averaging" [3] has been shown to be effective in locating coherent sources with a uniform linear array. At present, no such method is known for unequally spaced linear arrays.

Extension of other high resolution spectral estimation techniques such as maximum entropy, AR, and ARMA modeling, etc., to unequally spaced linear and/or two-dimensional arrays is the subject of much recent research [11, 12]. We plan to study and evaluate some of these algorithms during the coming year.

B. Array Geometry

The use of unequally spaced, small linear arrays is motivated primarily by the desire to avoid the grating lobes which result from uniform element spacing*. Elements spaced uniformly d wavelengths apart can determine the direction $u = \sin\theta$ of a source only modulo $\frac{\lambda}{d}$. This results from the fact that phase can only be determined modulo 2π and consequently sources at any two of the directions $u_n = \sin\theta_o \pm n\frac{\lambda}{d}$ produce identical amplitude and phase patterns at the array elements. This is no longer true when the spacing is irregular.

---

†Here again, we assume that the candidate direction vectors are linearly independent (i.e, the direction matrix V has full rank).

*Unequal spacing is also used as a form of tapering in large arrays, but these are not of interest to us.

An undesirable effect of unequal spacing is an increase in sidelobe level. Figure 3.1 shows the spacing of the four-element, 12.1λ QUICK LOOK array. Figure 3.2 shows the corresponding array factor (uniform weighting). While there are no perfect grating lobes, there are several extremely high sidelobes. The performance bounds obtained in Section II show that the sidelobes of the array factor are very important in determining the direction-finding capability of an array.

C. Linear Dependence of the Direction Vectors and Array Ambiguities

The study of the Ziv-Zakai bound leads one to the conclusion that when direction vectors are nearly linearly dependent, estimation accuracy suffers. In this section, the consequences of linear dependence are discussed, and a way of testing an array for this condition is proposed.

Suppose there exists some set of K direction vectors which are linearly dependent, i.e., there exist complex constants α_k such that

$$\sum_{k=1}^{K} \alpha_k \underline{v}(u_k) = 0$$

This means that the array is completely blind to this particular distribution of sources. Of course, the proper combination of complex amplitudes is required in order to get complete cancellation; if the sources are incoherent, this situation would not persist over multiple snapshots. However, since any one of the signals can be expressed as a linear combination of K-1 others, a distribution of K sources with <u>arbitrary</u> amplitudes cannot be distinguished from any one of K distributions of K-1 sources.

A simple example of this is a uniform linear array with element spacing λ. Such an array has grating lobes at u=±1. A signal arriving from direction u is indistinguishable from one arriving from directions u±1 (one of which is in "visible space"). These signals are linearly dependent. An estimator faced with this situation (and no a priori information) can do no better than to choose one of the two possible directions at random. The result is a large mean squared estimation error.

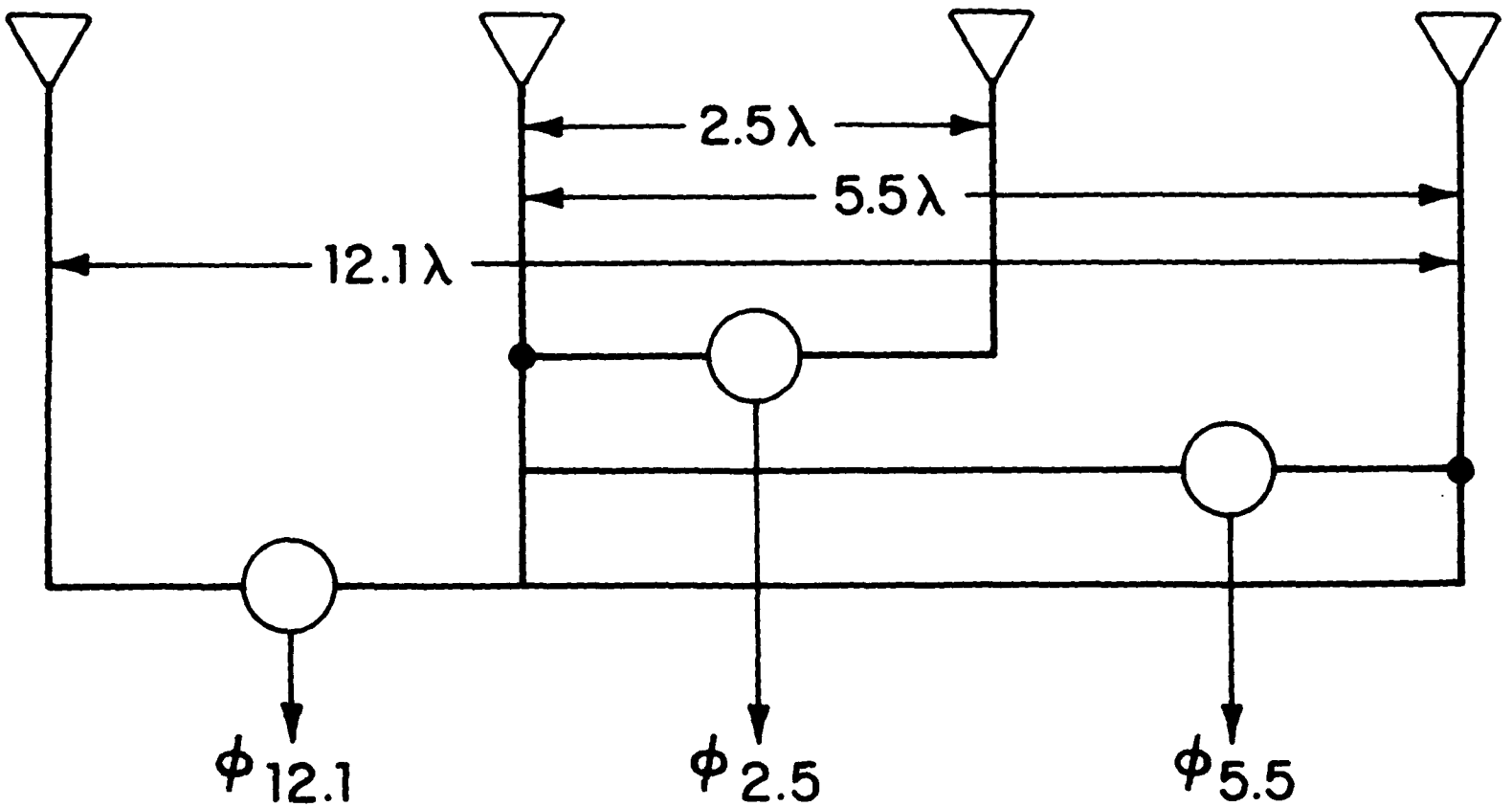


Fig. 3.1. Quick-look array.

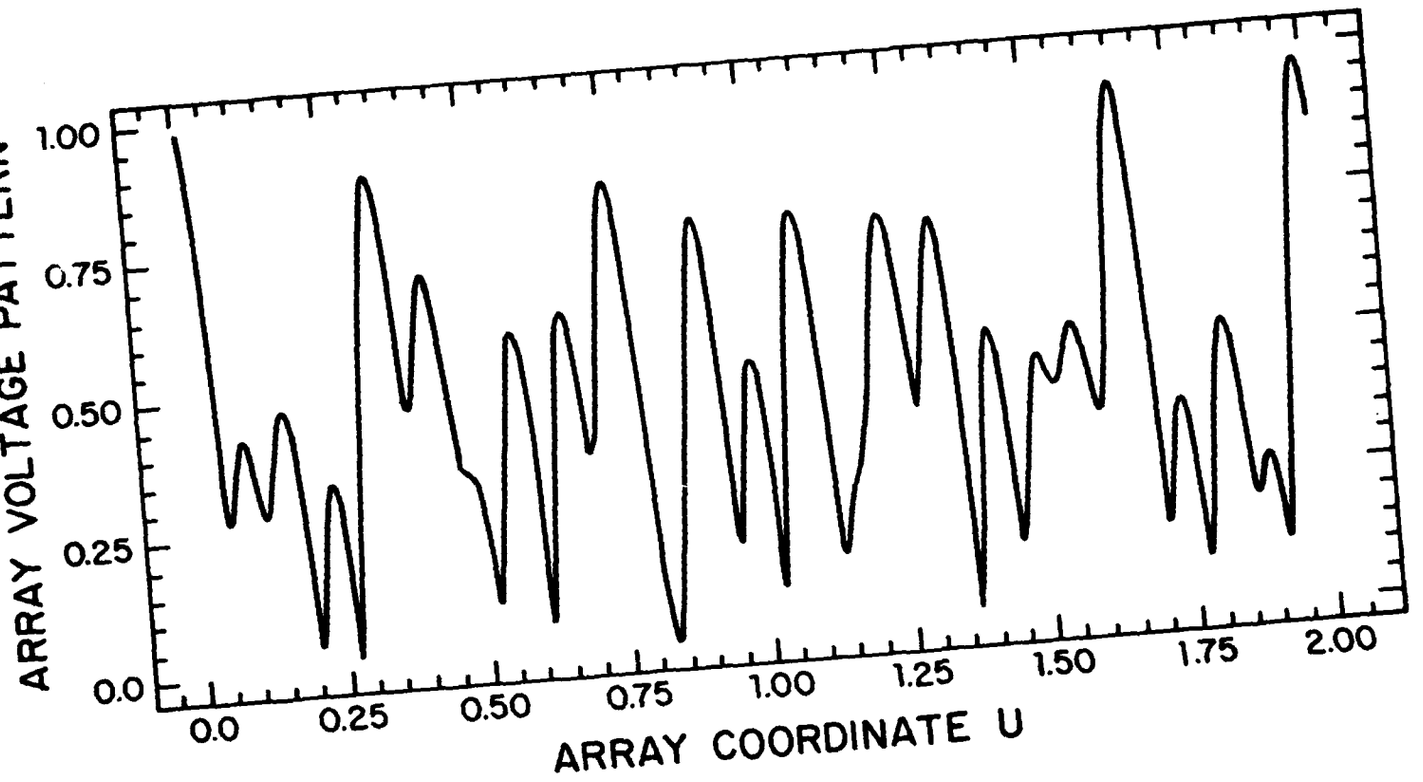


Fig. 3.2. Array pattern of quick-look array.

It can be shown that for equally spaced linear arrays without grating lobes, every set of distinct direction vectors is linearly independent. This result does not extend to nonequally spaced arrays. The following argument shows why.

Consider an M-element equally spaced linear array illuminated by L<M sources equally spaced in angle in the unit circle, i.e., $u_\ell = \frac{\lambda}{d}\frac{\ell}{L}$, $\ell = 1, ..L$. The response of array element m to source ℓ is

$$x_m(\ell) = \alpha_\ell e^{j2\pi\frac{m\ell}{L}}$$

where α_ℓ is the complex amplitude of source ℓ. The response of element m to all sources is

$$x_m = \sum_{\ell=1}^{L} x_m(\ell) = \sum_{\ell=1}^{L} \alpha_\ell e^{j2\pi\frac{m\ell}{L}}$$

Assume $\alpha_\ell = 1$. Since $e^{j\frac{2\pi}{L}}$ is an L^{th} root of unity, it follows that

$$x_m = \begin{cases} 0 & m \neq 0 \mod L \\ L & m = 0 \mod L \end{cases}$$

If the elements whose indices are 0 mod L are removed from the array, the remaining elements all have zero response, and the source distribution cannot be detected. This is just another way of saying that there exists a linear combination of direction vectors (sources) which is identically zero. The elements to be removed can be varied by adjusting the relative phases of the sources.

In order to cause problems, the linearly dependent source directions must all be in "visible space". The L sources span an interval in u of $\frac{\lambda}{d}\frac{L-1}{L}$; this length must be less than 2 if all sources are to be in visible space. Thus if the element spacing of the array satisfies $d > \frac{\lambda}{2}(1-\frac{1}{L})$ ambiguity problems will occur.

As a simple example, consider a five element array with spacing $d=\frac{\lambda}{2}$, and three sources at $u=0, \pm\frac{\lambda}{3d}$. The phase shift per element across the array is 0°, ±120° respectively for the three sources. Figure 3.3 shows how the complex amplitudes of the sources add up at each array element when they are in phase at the center element. When the center element is missing, the array cannot distinguish between one signal at broadside and two signals of the same amplitude at $u=\pm\frac{2}{3}$ both having 180° relative phase.

More generally, since for $u_1=0$, $u_2=e^{-j2\pi/3}$, $u_3=e^{j2\pi/3}$

$$\underline{v}_1 + \underline{v}_2 + \underline{v}_3 = 0$$

any combination of three sources from these three directions

$$a\underline{v}_1 + b\underline{v}_2 + c\underline{v}_3$$

cannot be distinguished from pairs of sources

$$(a-c)\underline{v}_1 + (b-c)\underline{v}_2$$
$$\text{or} \quad (a-b)\underline{v}_1 + (c-b)\underline{v}_3$$
$$\text{or} \quad (b-a)\underline{v}_2 + (c-a)\underline{v}_3.$$

If no information about the signal-in-space covariance matrix P is available, these ambiguities are fundamental; there is no way to determine the true source directions. However, if the signals are known a priori to be incoherent (diagonal P matrix) the true source directions <u>can</u> be determined. The reason is that only one of the possible configurations of sources will yield a diagonal P matrix.

An algorithm incorporating this feature is currently under investigation.

Testing for Linear Dependence

A set of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k$ is linearly dependent if and only if the matrix $G=V^HV = [\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_k]^H\ [\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_k]$ is singular, i.e., if $|V^HV|=0$.

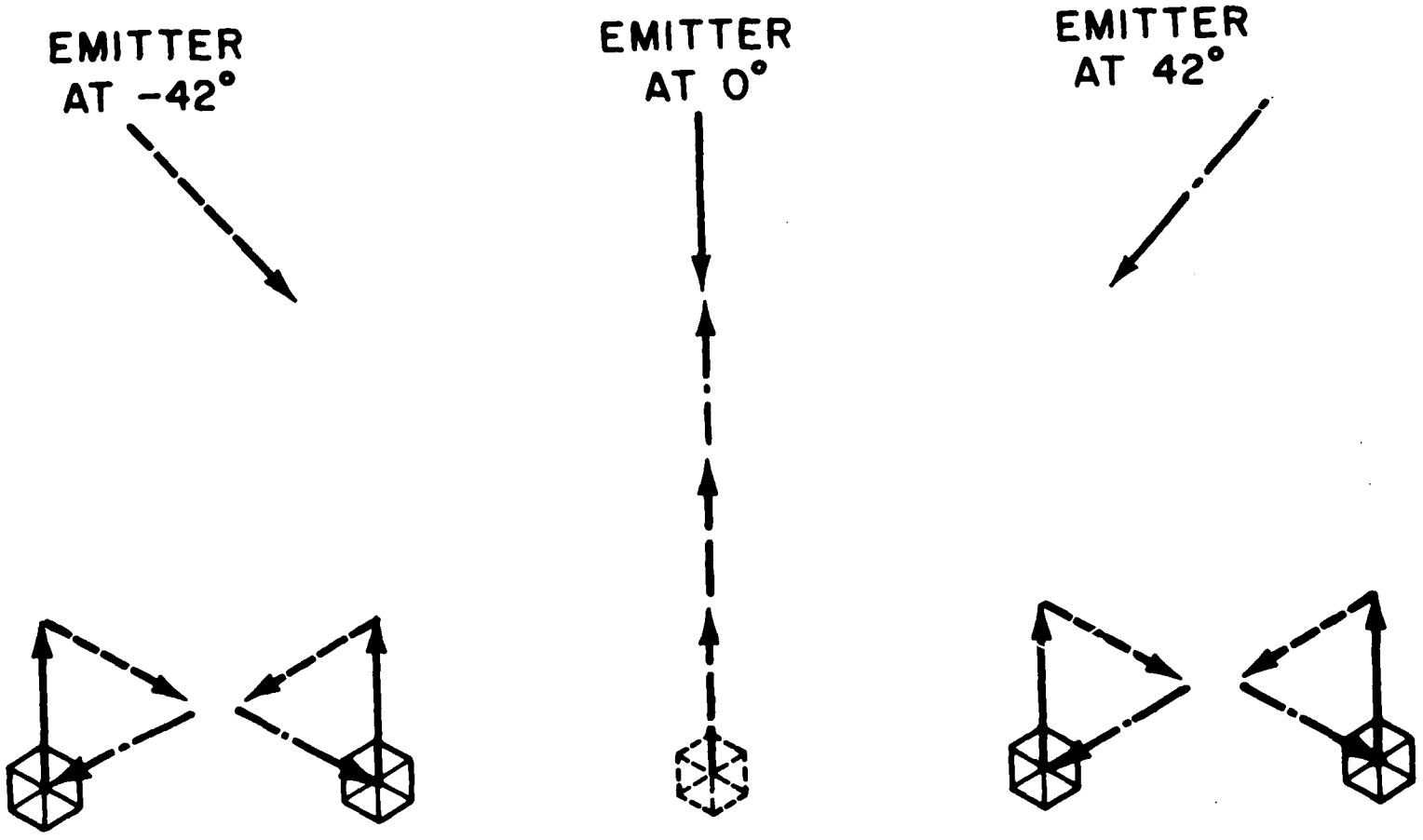


Fig. 3.3. Example of linearly dependent direction vectors: expurgated linear array.

The test for linear dependence is simply to evaluate the determinant as a function of the direction parameters $\underline{u}$ and look for zeros. Since $0 \le |V^H V| \le 1$, * an equivalent procedure is to look for peaks of $1 - |V^H V|$. For arrays of isotropic elements and k=2 signals, this function is simply the magnitude of the normalized array pattern, and the peaks represent grating lobes.

For k=3 signals on an isotropic array, we have

$$1-|V^H V| = 1-\begin{vmatrix} 1 & f(u_2-u_1) & f(u_3-u_1) \\ f^*(u_2-u_1) & 1 & f(u_3-u_2) \\ f^*(u_3-u_1) & f^*(u_3-u_2) & 1 \end{vmatrix}$$

where $f(u) = \frac{1}{M}\sum_{m=1}^{M} e^{j2\pi x_m u/\lambda}$ is the array pattern. This is a function of two *independent* variables, $x=u_2-u_1$, and $y=u_3-u_2$. Thus

$$G(x,y) = |f(x)|^2+|f(y)|^2+|f(x+y)|^2-2\mathrm{Re}\{f(x)f(y)f^*(x+y)\}$$

Without loss of generality, we may assume $u_1 \le u_2 \le u_3$, which implies $x, y \ge 0$. We are only interested in source distributions which are entirely in "visible space", i.e., $u_3 - u_1 = x+y \le 2$. Thus the region of interest becomes $0 \le y \le 2-x$, $0 \le x \le 2$. Furthermore, it follows immediately from the relation $f(-u) = f^*(u)$ that $G(y,x) = G(x,y)$. We plot it over the entire region for esthetic reasons.

Figure 3.4 shows a contour plot of the function G(x,y) over this region for the expurgated linear array of Fig. 3.3 with element spacing $\lambda/2$. The contours go from 0.5 to 1 in steps of 0.05. For any linear array, the direction vectors are dependent when two of the directions are equal. This creates ridges of ambiguity along the lines $u_2-u_1=0$ and $u_3-u_2=0$, the left side

---

* This assumes an ideal array whose direction vectors satisfy $\underline{v}^H(u)\underline{v}(u)=1$.

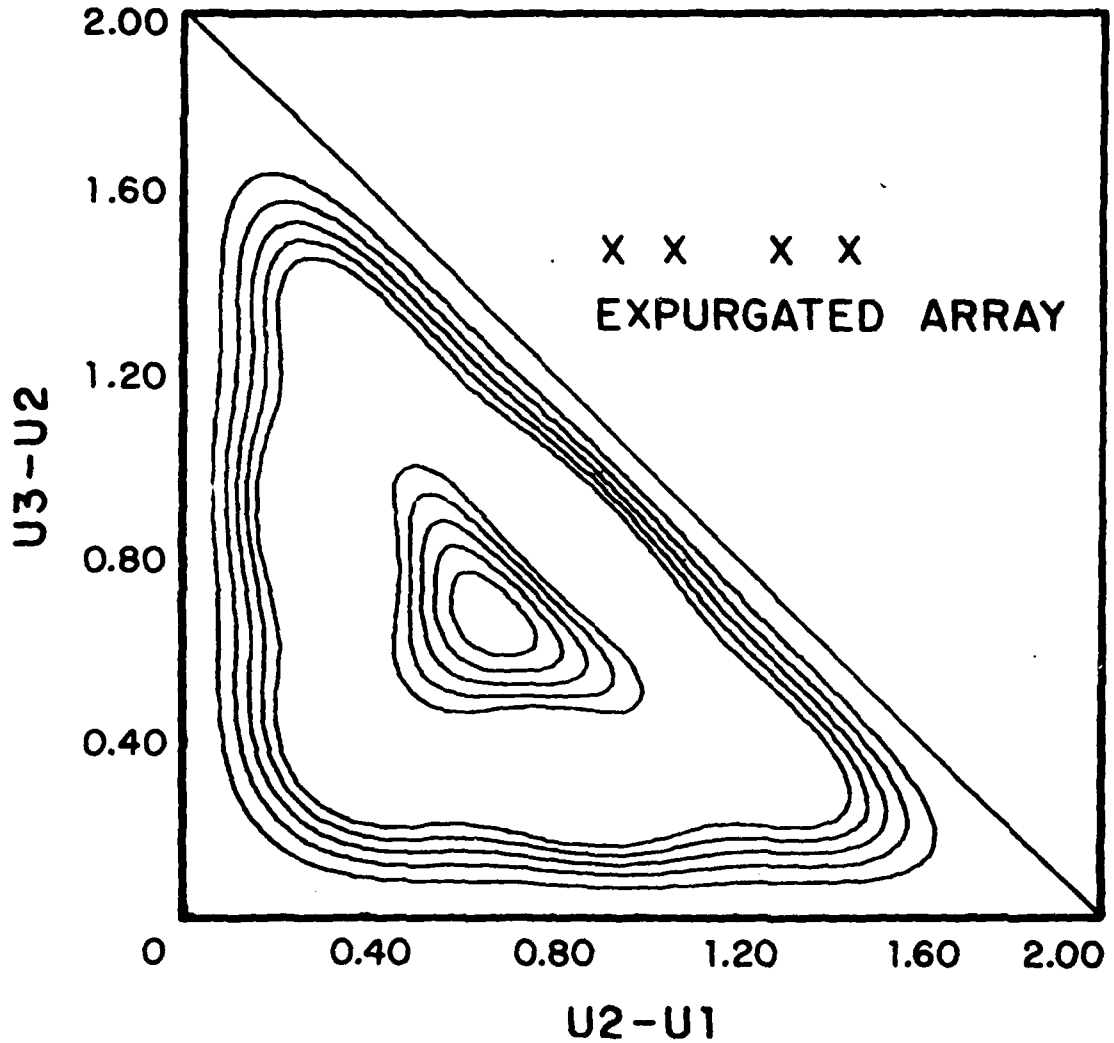


Fig. 3.4. G(x,y) for expurgated linear array.

and i from of the triangle. Also, for linear arrays (expurgated or not) whose element spacings are multiples of $\lambda/2$, there is a ridge of ambiguity along the hypotenuse of the triangle, which corresponds to $u_3=u_1+2$. The plot shows there is also ambiguity at x=y=2/3 which corresponds to the source distribution previously discussed.

The corresponding plot for the unexpurgated array is shown in Fig. 3.5. As expected, the ambiguity at x=y=2/3 has disappeared.

Somewhat more interesting results are obtained for the QUICK LOOK array. Figure 3.6 is an ambiguity plot for this array. The contour levels begin at 0.7 and go upward in steps of 0.05. The contours of high ambiguity are predominantly straight lines, either horizontal (constant $u_3 - u_2$), vertical (constant $u_2 - u_1$), or sloping downward to the right (constant $u_3 - u_1$). Close examination shows that the peaks (excluding those along the line $u_3 - u_2 = 0$) never reach unity. The largest peak, located at $u_2-u_1 = u_3-u_2 = 0.33038$, has value 0.99990.

The nature of Fig. 3.6 can be largely explained on the basis of the array pattern. This pattern is shown in Fig. 3.7 with the locations and values of the 9 largest sidelobes shown. Lines representing the sidelobe separations have been drawn on the figure. The ridges of high ambiguity follow these lines and their intersections produce the largest peaks. For example, there is a peak value of 0.997 at $u_2-u_1=1.65$, $u_3-u_2=0.33$. This corresponds to $u_1=-1.65$, $u_2=0$, $u_3=0.33$; signals in the main lobe and two large sidelobes.

The ridges represent near linear dependence between two direction vectors. At their intersections all three vectors are nearly equal. In addition, there are additional peaks, such as the one at $u_2-u_1=u_3-u_2=0.67$, where a linear combination of the three vectors in nearly zero, but no linear combination of two is.

To predict the location of these near ambiguities note that the QUICK LOOK array spacing is nearly 3, 2.5, 6.5λ. This is an expurgated linear array with $\frac{d}{\lambda} = 0.5$ and element spacing 6, 5, 13 d. Suitably phased equal amplitude sources at $\frac{d}{\lambda} u=0, \pm \frac{1}{3}$ produce outputs on the array at element numbers 3k+1, none of which are present; consequently, these direction vectors

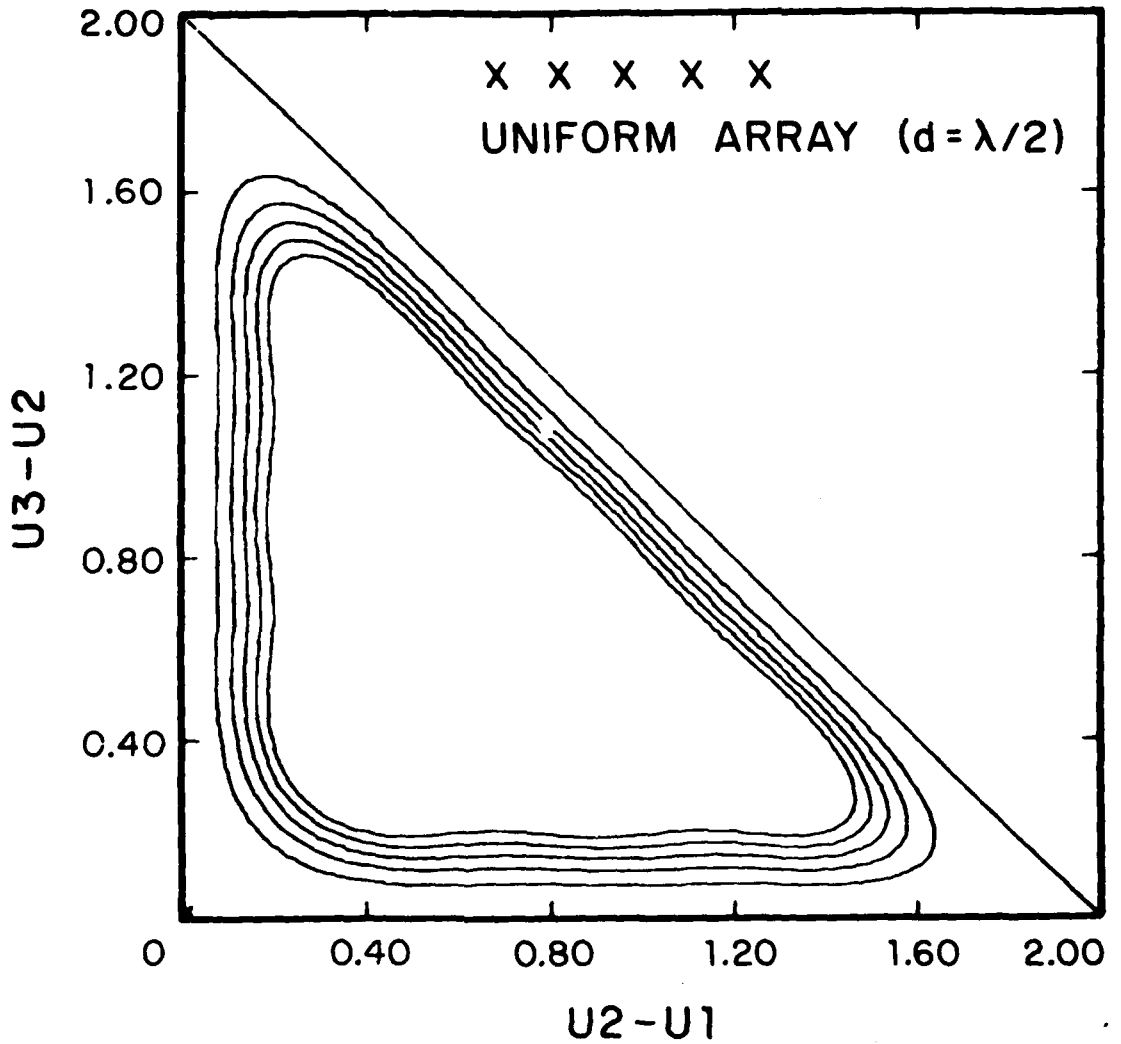


Fig. 3.5. G(x,y) for uniform linear array.

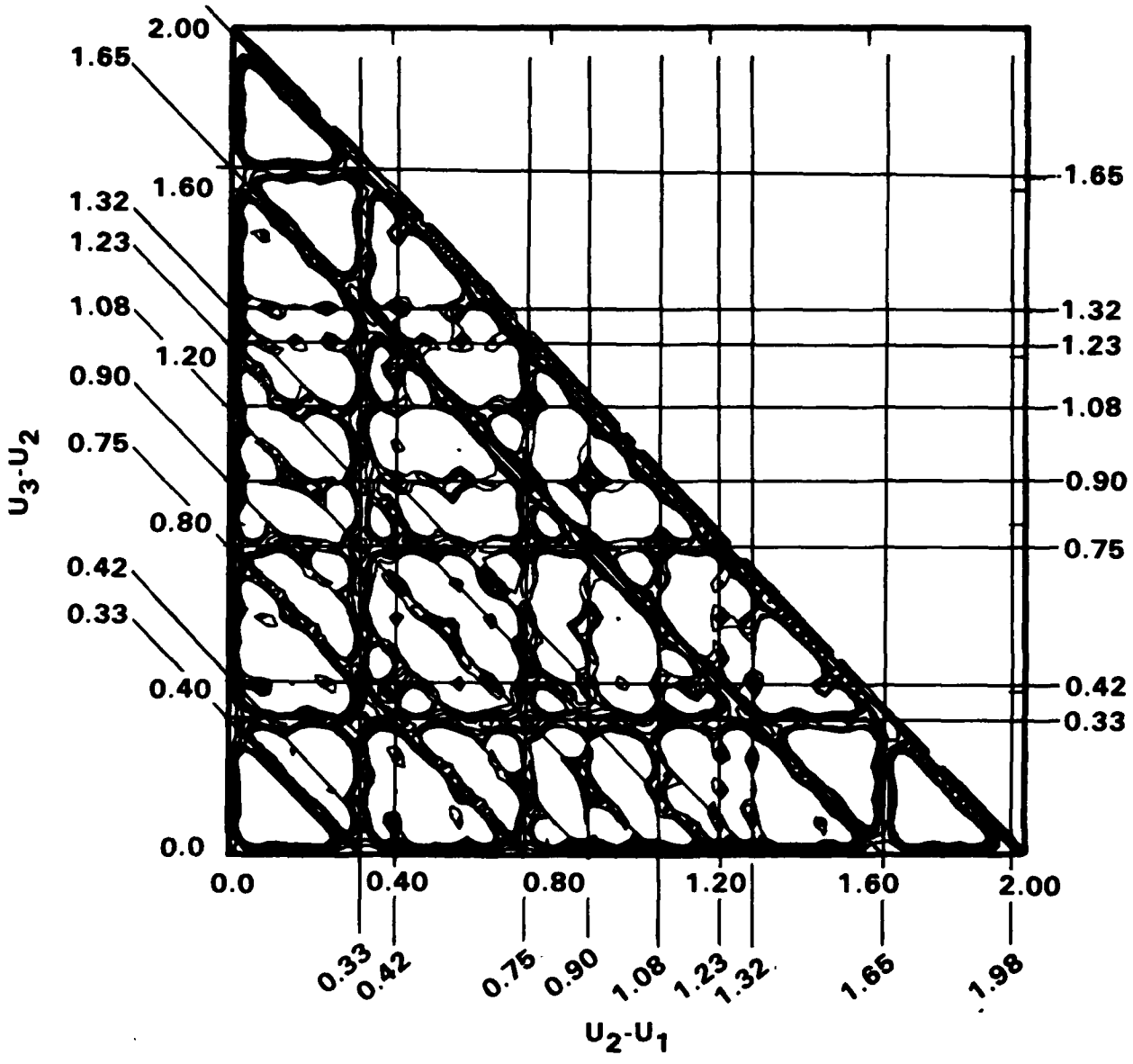


Fig. 3.6. G(x,y) for QUICK-LOOK array.

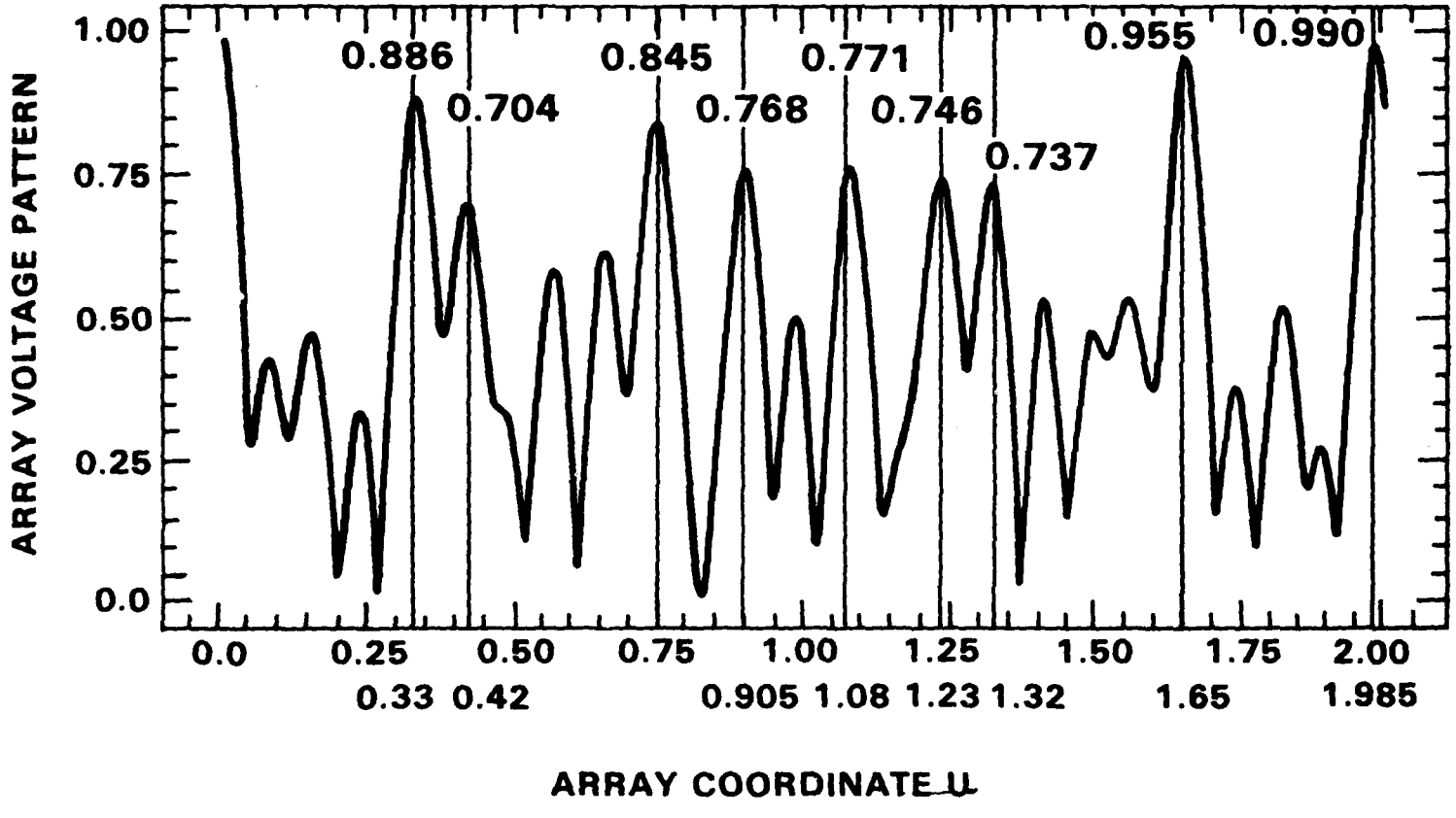


Fig. 3.7. Array pattern of QUICK-LOOK array.

are linearly dependent. Although certainly not obvious at the outset, an ambiguity also exists for $\frac{d}{\lambda}\,u=0,\ \pm\frac{1}{6}$, viz.

$$\begin{pmatrix}1\\1\\e^{j\pi/3}\\1\end{pmatrix} - (1+e^{j\pi/3})\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + e^{j\pi/3}\begin{pmatrix}1\\1\\e^{-j\pi/3}\\1\end{pmatrix} = 0$$

The array is also nearly 3, 2.4, 6.6λ, which is an expurgated linear array with $\frac{d}{\lambda}=0.6$ and element spacing 5, 4, 11d. Sources at $u=0$, $\frac{\lambda}{d}\left(\frac{1}{3}+\ell\right)$, $\frac{\lambda}{d}\left(\frac{2}{3}+m\right)$ for any integers ℓ, m will produce zero input. An example that falls within the region plotted is $u_2-u_1=u_3-u_2 = 5/9$.

A third approximation to the array is 3.3, 2.2, 6.6λ, with $d/\lambda = 1.1$ and spacing 3, 2, 6d. This produces ambiguities at $u_2-u_1=u_3-u_2=0.303$, 0.606, 0.909 at $u_2-u_1=1.212$, $u_2-u_1=1.212$, $u_3-u_2=0.303$, and perhaps elsewhere.

Since these are only approximations to the QUICK LOOK array, its ambiguities are not perfect. However, all produce ambiguity peaks which exceed 0.95.

The fact that the ambiguities of the array are not perfect means that at sufficiently high S/N ratio, the correct source distribution can be determined. However, in near ambiguous situations, the required S/N can be very large. This can be shown via the Ziv-Zakai bound for two signals (see Appendix VII).

Array ambiguities involving more than three directions also exist. For N directions, the quantity to be examined is a function of N-1 variables, and finding the peaks is a much more laborious process. For this reason, we confine our attention to ambiguities involving only three directions.

This study of linear dependence of direction vectors has led to the following conclusions for unequally spaced arrays.

1. Ambiguous source distributions involving combinations of two or more sources exist for expurgated linear arrays.

2. For more general unequally spaced arrays, these ambiguities are usually not perfect. However, the threshold signal-to-noise ratio necessary to resolve them can be quite large.

IV. QUICK LOOK Performance Simulation

A. One Emitter

QUICK LOOK performance was evaluated first for the case of a single emitter. Theory predicts and simulations confirm that at high S/N ratio, the maximum likelihood estimate of $u = \sin\theta$ is asymptotically efficient, i.e., approaches the Cramer-Rao bound, and thus is the best possible (unbiased) estimate. The ML estimate is just the value of u for which the steered array output

$$\left| \sum_{m=1}^{M} z_m \, d^{-j2\pi \frac{x_m}{\lambda} u} \right|$$

is a maximum. This is referred to here as beamsum processing.

Current QUICK LOOK processing, which assumes only one emitter is present, uses only phase differences between three pairs of elements. This processing was also simulated. A correction to the algorithm, detailed in Appendix D, was found to be necessary in order to get correct results for emitters near end fire.

Figure 4.1 shows simulation results for beamsum processing for emitters at $u = 0, 0.5, .855, .995$ ($\theta = 0, 30°, 60°, 84°$). The RMS error in beamwidths* is plotted vs. array signal-to-noise ratio for each case. Also shown for comparison is the single emitter Cramer-Rao bound. The threshold SNR, where the error increases abruptly, depends on the emitter direction. This is due to the nature of the array pattern of the QUICK LOOK array (Fig. 3.7), as we now show.

---

*Beamwidth is defined to be $\frac{M-1}{M} \frac{\lambda}{L}$ where M is the number of array elements and L is the array length.

Beamsum processing scans the beam over all of "visible space," i.e., from $u = -1$ to $u = 1$, and takes the direction of maximum response as the emitter direction. When the emitter is at broadside (0°), the result of the scan is a noisy version of the array pattern from $u = -1$ to $u = 1$ (the array pattern is symmetric abut $u = 0$). The largest sidelobe in this interval is 0.88535 at $u = \pm 0.33$. When the SNR drops sufficiently so that these three peaks can be confused, the direction-finding error starts to increase abruptly. When the emitter is at 30°, the region of interest is $u = -1.5$ to $u = 0.5$. Since the largest sidelobe is unchanged, the threshold SNR remains nearly the same. With an emitter at 60°, however, the region of interest is $u = -1.866$ to $u = 0.134$, and the largest sidelobe is 0.95467 at $u = -0.784$. The threshold SNR should, therefore, increase by about $(1-0.88636)^2/(1-0.95467)^2$ or about 8 dB. An emitter at 84° brings an even larger sidelobe with height 0.96962 into the visible region. The resulting change in threshold SNR should be about 3.5 dB. These predictions are in reasonable agreement with the simulation results.

The ZZ worst-case bound (2.33) does a reasonable job of predicting the threshold SNR. Figure 4.2 shows again the simulation results for beamsum processing with a single emitter at 0° or at 60°, together with the appropriate ZZ bound. For example, when the emitter is at 60°, the appropriate ZZ bound is the worst-case value taken over $u = -1.866$ to $u = 0.134$.

The QUICK LOOK processing algorithm is not a pure estimator but combines estimation and detection (see Appendix VIII). When phase ambiguity resolution fails, the algorithm rejects the data and makes no estimation of direction. At low S/N (< 10 dB), only about 20% of the trials result in estimates. Such "edited" estimates are not subject to the performance bounds discussed previously, except in the limit of large S/N.

For this reason, the QUICK LOOK algorithm was modified for purposes of comparison with other pure estimators and bounds. When ambiguity resolution fails, the modified algorithm chooses a random number between -1 and 1 for the emitter direction u. Simulation results for this modified algorithm are shown

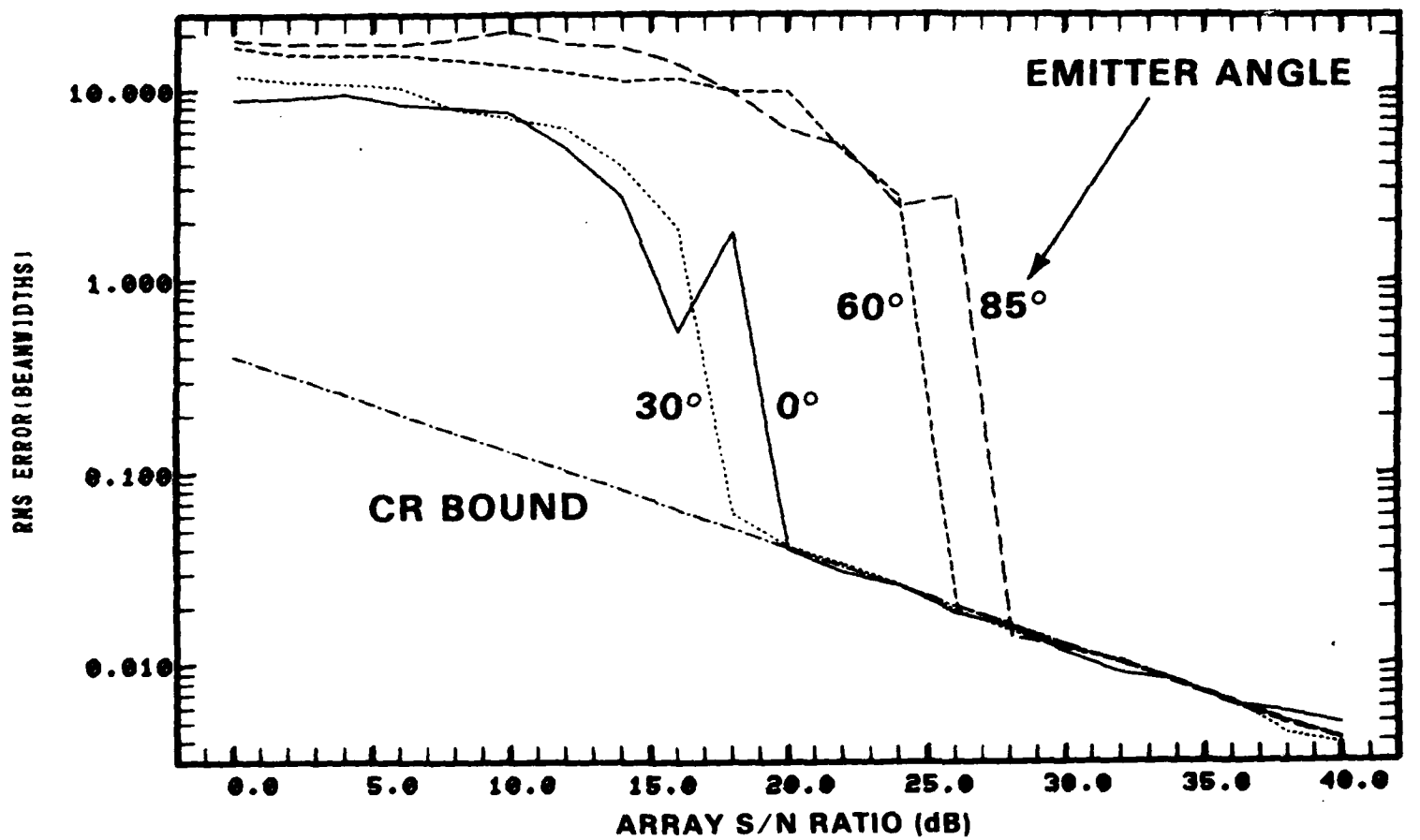


Fig. 4.1. Single emitter simulation results for QUICK-LOOK array with beam-sum processing.

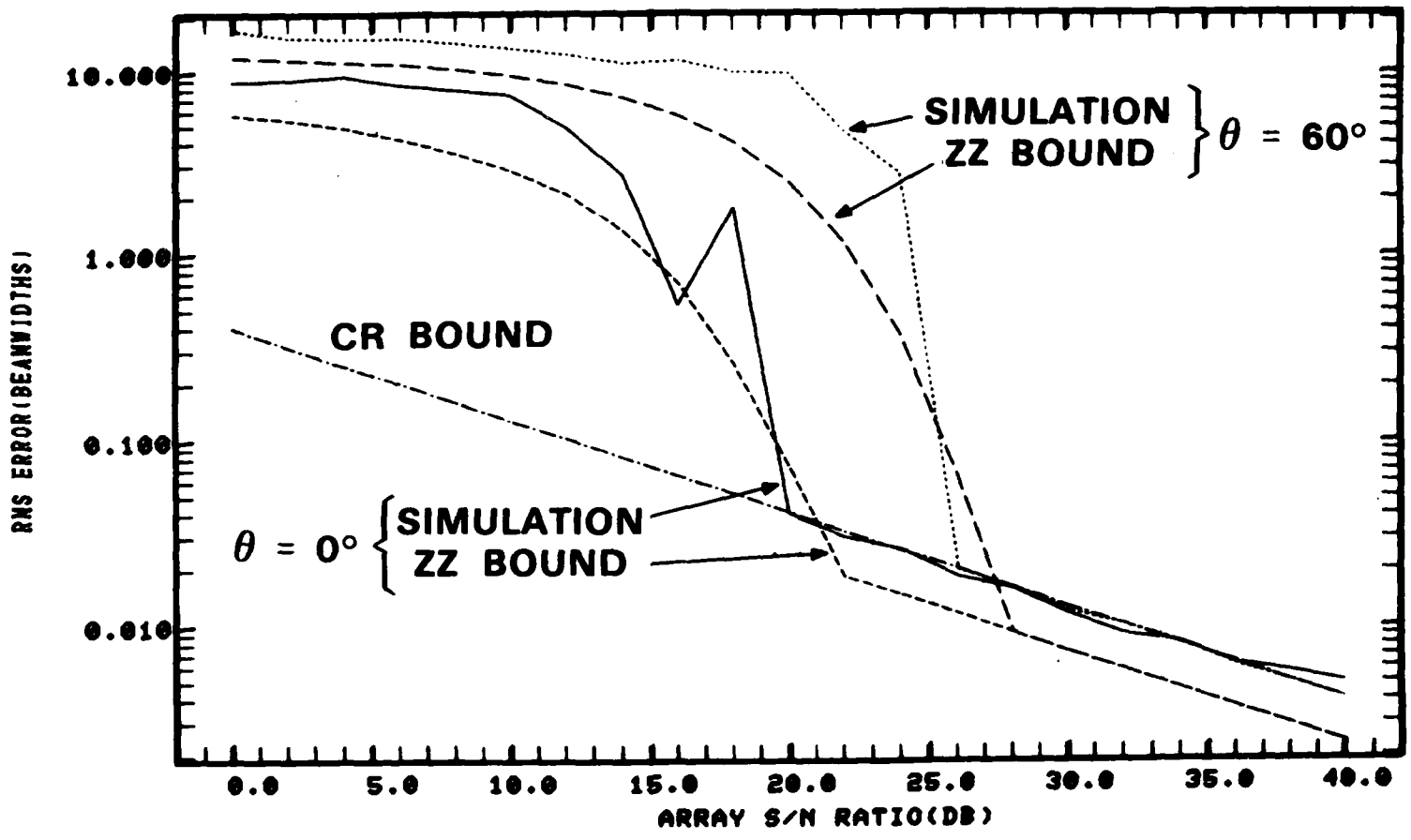


Fig. 4.2. Comparison of beam-sum processing with Ziv-Zakai bounds.

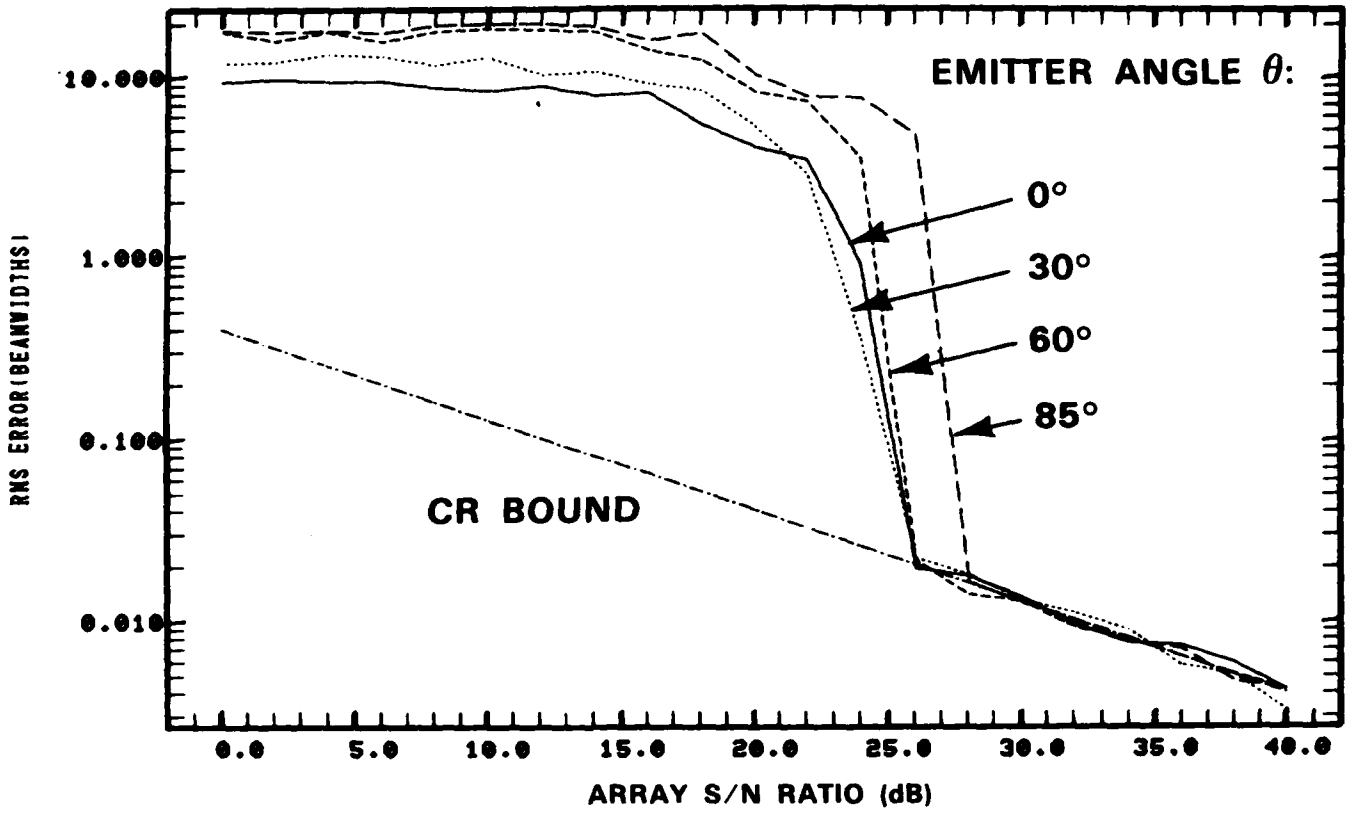


Fig. 4.3. Single emitter simulation results for QUICK-LOOK array with QUICK-LOOK (phase only) processing.

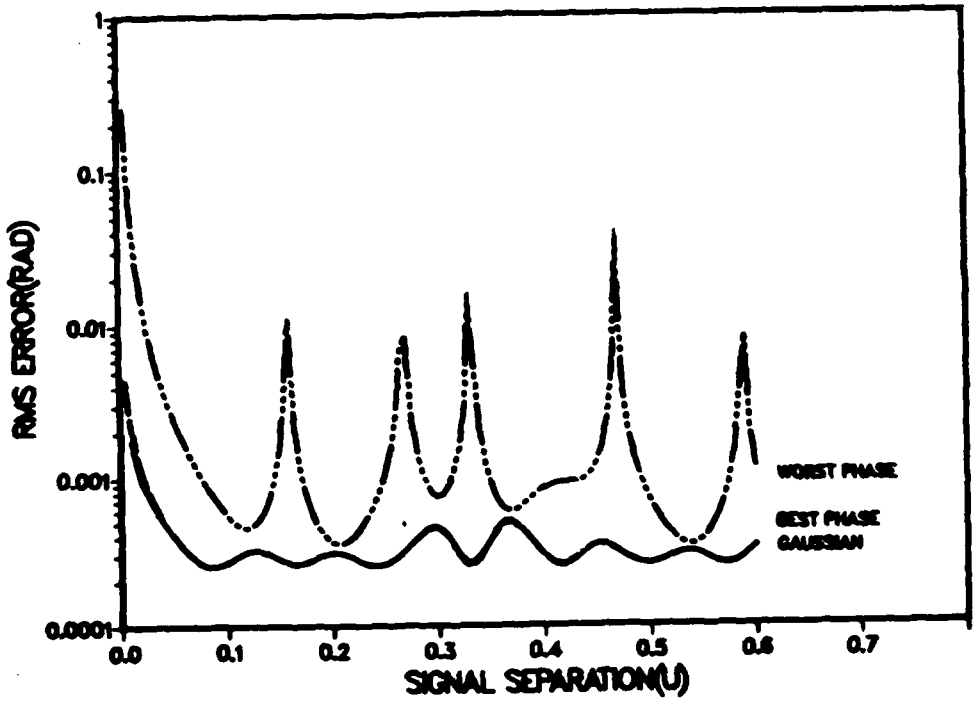


Fig. 4.4. Two-emitter Cramer-Rao bounds for Gaussian and deterministic signal models.

in Fig. 4.3 together with the CR bound. At high SNR, the algorithm achieves the CR bound; its threshold SNR is somewhat higher, particularly for emitters near broadside.

The most important conclusions to be drawn from the simulations are:

(1) Beamsum processing offers little improvement in the single-emitter case. The (modified) QUICK LOOK algorithm is essentially optimum.

(2) Ziv-Zakai bounds can be used to determine a bound on threshold S/N for estimating the direction of arrival of a single signal.

B. Two Emitters

Two emitter CR bounds on the standard deviation of the angular error* are shown as functions of emitter separation in Fig. 4.4. Plotted are the bounds for a Gaussian signal model and for a deterministic signal model with best and worst case phase relationships. The received signals have equal power (20 dB array S/N) and 100 observations are available. In the deterministic model, the phase difference between the signals is a critical parameter. The difference in performance between best and worst case phase conditions is often appreciable. However, it was pointed out in section II.1.E that when multiple observations are available and the signal phases are random, the correlation between the directional errors will be small and the deterministic and Gaussian signal models yield similar bounds on performance. For these reasons, the Gaussian signal model was used in the simulation.

In the single emitter simulation, the maximum likelihood (ML) algorithm (beamsum) was used. For the two-emitter simulation, the MUSIC algorithm [10] was used.

Figure 4.5 shows the results for signal separations greater than 1 beamwidth. Each point represents the standard deviation in the angle error for a particular emitter after 100 Monte Carlo trials. Two emitters of equal

---

*More precisely, the standard deviation of the error in $u = \sin\theta$.

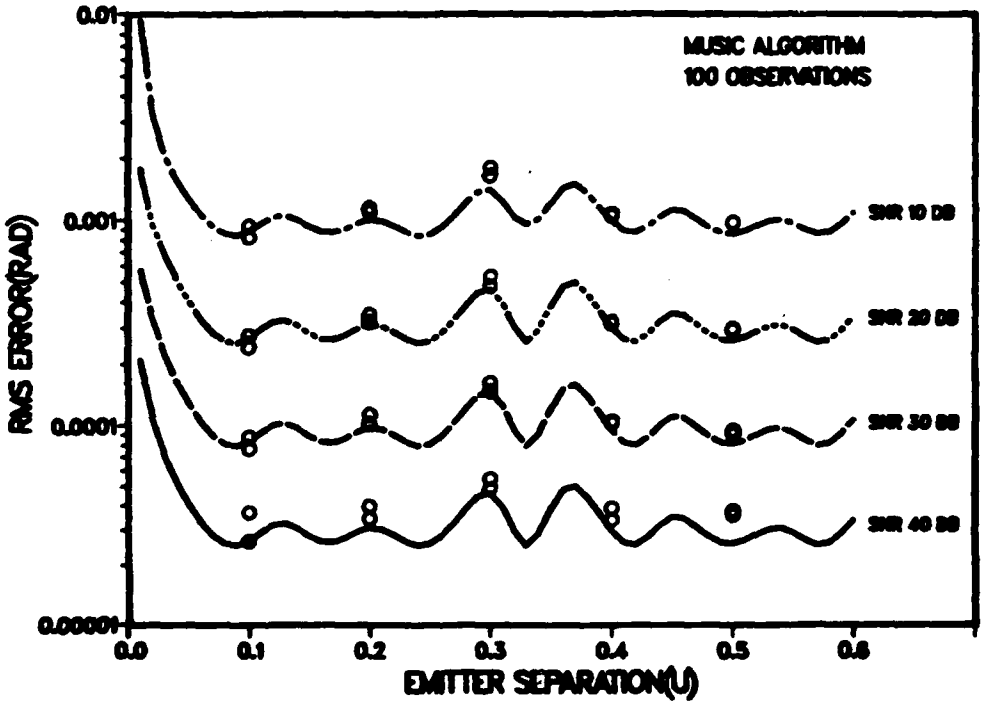


Fig. 4.5. Two-emitter simulation results and CR bounds for QUICK-LOOK array (emitter separation > beamwidth).

power comprised the simulated signal, so two points were produced for each set of 100 trials. Signal separations of 0.1 to 0.5 in $u = \sin\theta$, and S/N ratios of 10 to 40 dB were simulated. Shown also are the CR bounds (Gaussian signal model). The agreement is very good, indicating that the MUSIC algorithm achieves nearly optimal performance for signal separations in excess of 1 beamwidth.

Fig. 4.6 shows results for signal separations of 0.05, 0.1, 0.2, 0.4, and 0.8 beamwidths. A breakdown in estimation accuracy occurs as the emitters come closer together in angle. The mechanism is that the emitters are no longer resolved; they produce only one peak in the spectrum. Consequently a spurious peak is accepted and its angle estimate assigned to one of the two emitters, causing a large RMS error. A similar effect was noted in [3] where a two emitter simulation was run using the MUSIC algorithm and a uniform linear array.

In the absence of a Ziv-Zakai bound for the two emitter case, it is not possible to say on the basis of the present results whether or not the resolution performance of QUICK LOOK can be improved by the choice of a different high resolution algorithm. Simulations using other algorithms are now being conducted to resolve this question.

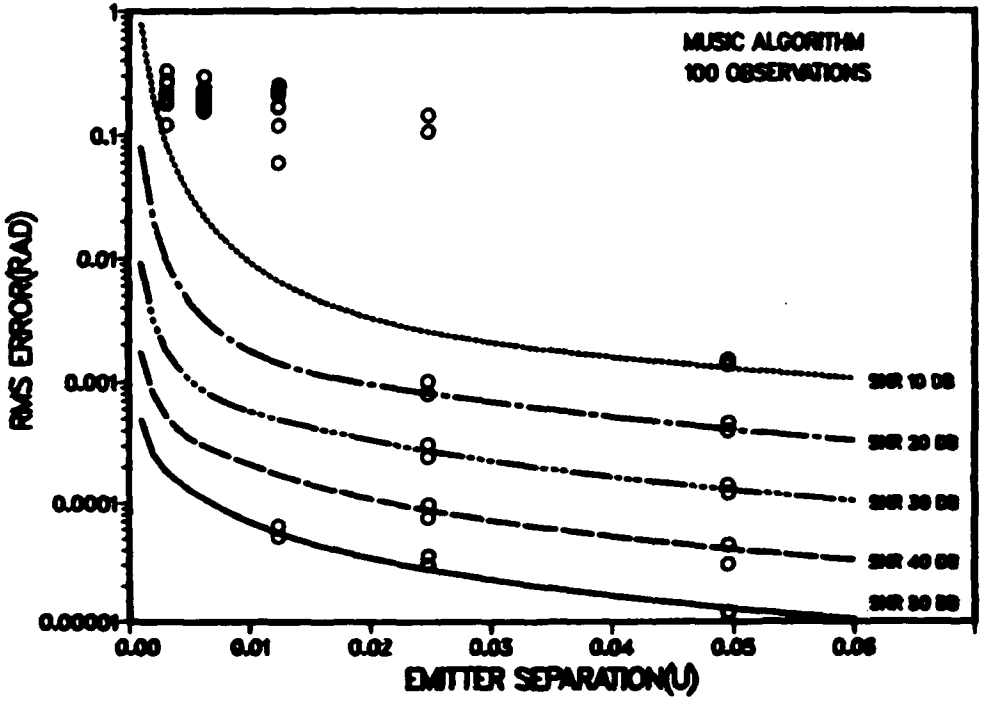


Fig. 4.6. Two-emitter simulation results and CR bounds for QUICK-LOOK array (emitter separation < beamwidth).

V. Summary

The problem of evaluating the emitter location capability of arrays has been addressed by studying Cramer-Rao and Ziv-Zakai bounds on angle estimation accuracy. These bounds are useful for two reasons:

1. They provide a standard against which angle estimation algorithms can be judged.
2. They permit comparisons of various array geometries which are algorithm-independent.

Although not leading directly to an analytical theory of optimum array geometries, the study clarified the notion of array ambiguities in nonuniformly spaced linear arrays and revealed a method for finding them. A power estimation technique for resolving these ambiguities when the signals are incoherent is now being tested.

The study also showed the connection between high array sidelobes and the threshold *signal-to-noise* ratio required to locate one emitter. When one sidelobe is dominant, threshold S/N is directly related to the height of the sidelobe; this becomes less true if multiple sidelobes of roughly equal height are present. Nevertheless, a useful rule of thumb for reducing the required S/N is to minimize the maximum sidelobe of the array.

Special attention was devoted to simulating the performance of the QUICK LOOK array. For a single emitter, QUICK LOOK processing was found to be essentially optimum. For two emitters, the lack of an appropriate Ziv-Zakai bound makes it difficult to say whether the breakdown in emitter resolution capability is a property of the algorithm employed or a fundamental limitation caused by the array geometry. This question is presently being resolved via analysis and simulation.

APPENDIX I

IMPROVED ZZ BOUND

Wax and Ziv [9] have improved the basic Ziv-Zakai bound by a factor of two. Their derivation assumes that an estimate $\hat{x}$ of some parameter $x \in [-A, A]$ takes values in the finite, symmetric interval $[-\hat{A}, \hat{A}]$ where $\hat{A} < A$. The result is then

$$\frac{1}{2}\left[e^2(a) + e^2(-a)\right] > 2\, a^2\, P_E(-a, a)$$

a bound on the average mean squared error when the true values are x=a and x=-a. This is not as specialized a result as it appears to be at first glance.

Given any parameter x on a finite interval [A, B] and any two values $x_1, x_2 \in [A, B]$, the linear transformation $y = x - \frac{x_1 + x_2}{2} = x - x_o$ maps x_1 into -a, x_2 into a, where $a = \frac{x_2 - x_1}{2}$. The new parameter y takes values on the non-symmetric interval $[A - x_o, B - x_o]$. We also use the transformation to generate a new estimate $\hat{y}$ taking values in this same interval. The mean squared error of the estimate is preserved under the transformation.

Now define a truncated estimate $\hat{y}_t$ on the largest symmetric interval contained in the mapping of [A, B], namely [-Y, Y], where

$$Y = \min\left(|A - x_o|, |B - x_o|\right)$$

If $\hat{y}$ lies outside of this interval, $\hat{y}_t = \pm Y$ as appropriate. The mean squared error of this truncated estimate will be smaller than that of $\hat{y}$ when the true values are x_1, x_2. Thus,

$$\frac{1}{2}\left[e^2_{\hat{x}}(x_1) + e^2_{\hat{x}}(x_2)\right] = \frac{1}{2}\left[e^2_{\hat{y}}(a) + e^2_{\hat{y}}(-a)\right]$$

$$> \frac{1}{2}\left[e^2_{\hat{y}_t}(a) + e^2_{\hat{y}_t}(-a)\right] > 2a^2 P_{E_{\hat{y}_t}}(-a, a) = \frac{(x_2 - x_1)^2}{2} P_{E_{\hat{x}}}(x_1, x_2)$$

This shows that the result holds for any finite interval, symmetric or not.

APPENDIX II

PROBABILITY OF ERROR IN CHOOSING BETWEEN SIGNALS OF KNOWN COMPLEX AMPLITUDE

The contending hypotheses are:

$$H_1: \underline{z} = p\,\underline{s}(u_1) + \underline{\xi}$$

$$H_2: \underline{z} = p\,\underline{s}(u_2) + \underline{\xi}$$

where $p = Ae^{j\phi}$ is the known complex amplitude,

$$|\underline{s}(u_1)|^2 = 1 \quad ,$$

and $\underline{\xi}$ is a complex Gaussian noise vector with zero mean and $E\,\underline{\xi}\,\underline{\xi}^T = 0$, $E\,\underline{\xi}\,\underline{\xi}^H = \Lambda$.

A decision rule is a division of the complex M-dimensional space a of all possible observation vectors $\underline{z}$ into two parts, a_1 and a_2. If the observation falls in a_1, choose H_1; otherwise, H_2. The a priori probabilities of the two hypotheses are P_1 and P_2.

The probability of error for this decision rule is

$$P_E = P_2 Pr\{\underline{z} \in a_1 | H_2\} + P_1 Pr\{\underline{z} \in a_2 | H_1\}$$

$$= P_2 \int_{a_1} p(\underline{z}|H_2)d\underline{z} + P_1 \int_{a-a_1} p(\underline{z}|H_1)d\underline{z}$$

$$= P_1 + \int_{a_1} [P_2 p(\underline{z}|H_2) - P_1 p(\underline{z}|H_1)]d\underline{z}$$

To minimize P_E, assign $\underline{z}$ to a_1 if and only if

$$P_1 p(\underline{z}|H_1) > P_2 p(\underline{z}|H_2)$$

An equivalent and more convenient form is

$$\ln p(\underline{z}|H_1) - \ln p(\underline{z}|H_2) \gtrless \ln \frac{P_2}{P_1}$$

This is the well-known likelihood ratio test. For the problem at hand,

$$p(\underline{z}|H_i) = \frac{1}{\pi^M |\Lambda|} e^{-|\underline{z} - p\underline{s}(u_i)|^2} \qquad i = 1,2$$

and the likelihood ratio test becomes

$$x \triangleq 2\mathrm{Re}\{p\underline{z}^H \Lambda^{-1}(\underline{s}_1 - \underline{s}_2)\} + A^2(\underline{s}_2^H \Lambda^{-1} \underline{s}_2 - \underline{s}_1^H \Lambda^{-1} \underline{s}_1) \gtrless_{H_2}^{H_1} \ln \frac{P_2}{P_1} = \lambda$$

x is a real Gaussian random variable with mean

$$m_1 = E(x|H_1) = A^2 |\underline{s}_1 - \underline{s}_2|_\Lambda^2{}^{-1}$$

under H_1 and

$$m_2 = E(x|H_2) = -m_1$$

under H_2. The variance of x, which is the same under either hypothesis, is given by

$$\sigma_x^2 = E(4\mathrm{Re}^2\{p\underline{\xi}^H \Lambda^{-1}(\underline{s}_1 - \underline{s}_2)\})$$

$$= 2A^2 |\underline{s}_1 - \underline{s}_2|_\Lambda^2{}^{-1} = 2m_1$$

The probability of error is

$$P_E = P_2 Pr\{x > \lambda | H_2\} + P_1 Pr\{x < \lambda | H_1\}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \left[P_2 \int_{\lambda}^{\infty} e^{-\frac{1}{2\sigma_x^2}(x + m_1)^2} dx + P_1 \int_{-\infty}^{\lambda} e^{-\frac{1}{2\sigma_x^2}(x - m_1)^2} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[P_2 \int_{\frac{\lambda+m_1}{\sigma_x}}^{\infty} e^{-\frac{x^2}{2}} dx + P_1 \int_{-\infty}^{\frac{\lambda-m_1}{\sigma_x}} e^{-\frac{x^2}{2}} dx \right]$$

$$= P_2 \text{erfc}_*\left(\frac{m_1 + \lambda}{\sigma_x}\right) + P_1 \text{erfc}_*\left(\frac{m_1 - \lambda}{\sigma_x}\right)$$

To get the tightest bound, we maximize this expression with respect to λ. Since

$$\frac{P_2}{P_1} = \frac{1 - P_1}{P_1} = e^{\lambda}$$

we have

$$P_1 = 1/(1 + e^{\lambda})$$

$$P_2 = 1/(1 + e^{-\lambda})$$

$$P_E(\lambda) = \frac{1}{1 + e^{-\lambda}} \text{erfc}_*\left(\frac{m_1 + \lambda}{\sigma_x}\right) + \frac{1}{1 + e^{\lambda}} \text{erfc}_*\left(\frac{m_1 - \lambda}{\sigma_x}\right)$$

The derivative is

$$\frac{dP_E}{d\lambda} = \frac{1}{2(1+\cosh\lambda)}\left[\operatorname{erfc}_*\left(\frac{m_1+\lambda}{\sigma_x}\right) - \operatorname{erfc}_*\left(\frac{m_1-\lambda}{\sigma_x}\right)\right]$$

$$+ \frac{1}{\sqrt{2\pi}\,\sigma_x}\left[\frac{1}{1+e^{\lambda}}\, e^{-\frac{(m_1-\lambda)^2}{2\sigma_x^2}} - \frac{1}{1+e^{-\lambda}}\, e^{-\frac{(m_1+\lambda)^2}{2\sigma_x^2}}\right]$$

When the expressions for m_1 and σ_x are substituted into this expression, the second term vanishes and the derivative becomes

$$\frac{dP_E}{d\lambda} = \frac{1}{2(1+\cosh\lambda)}\left[\operatorname{erfc}_*\left(\frac{m_1+\lambda}{\sigma_x}\right) - \operatorname{erfc}_*\left(\frac{m_1-\lambda}{\sigma_x}\right)\right]$$

This is obviously zero when $\lambda = 0$; furthermore, it is clearly always negative for $\lambda > 0$ and positive for $\lambda < 0$. Thus, $P_E(\lambda)$ has a single maximum at $\lambda = 0$, (equal a priori probabilities) and

$$\max_{\lambda} P_E(\lambda) = \operatorname{erfc}_* \frac{m_1}{\sigma_x} = \operatorname{erfc}_* \left(\frac{Ad}{\sqrt{2}}\right) \qquad \text{(II.1)}$$

where $d = |\underline{s}_1 - \underline{s}_2|_{\Lambda^{-1}}$ is the distance between the signals.

For multiple snapshots $\underline{z}_n$, n=1, N the likelihood ratio test becomes

$$x = \sum_{n=1}^{N} \left[2\mathrm{Re}\{p_n \underline{z}_n^* \Lambda^{-1}(\underline{s}_1 - \underline{s}_2)\} + |p_n|^2 (\underline{s}_2^H \Lambda^{-1} \underline{s}_2 - \underline{s}_1^H \Lambda^{-1} \underline{s}_1)\right] \underset{H_2}{\overset{H_1}{\gtrless}} 0$$

If the total signal power is defined to be

$$A^2 = \sum_{n=1}^{N} |p_n|^2$$

then the previous equations for the means and variance of x still apply and

the result (II.I) is still valid. The only effect of multiple snapshots on the bound is to increase the signal-to-noise ratio.

APPENDIX III

ERROR PROBABILITY FOR CHOOSING BETWEEN SIGNALS WITH UNKNOWN PHASE

The contending hypotheses are

$$H_1: \underline{z} = Ae^{j\phi}\ \underline{s}(u_1) + \underline{\xi}$$

$$H_2: \underline{z} = Ae^{j\phi}\ \underline{s}(u_2) + \underline{\xi}$$

where the statistics of $\underline{\xi}$ are as defined in Appendix II.

We use the result (II.1) of Appendix II, replacing $\underline{s}_i$ by $\underline{s}_i e^{j\phi i}$, and obtain immediately

$$P_E = \text{erfc}\ \left(\frac{Ad}{\sqrt{2}}\right) \qquad \text{(III.1)}$$

where now

$$d^2 = |e^{j\phi 1}\ \underline{s}_1 - e^{j\phi 2}\ \underline{s}_2|^2_{\Lambda^{-1}} = |\underline{s}_1 - e^{j\beta}\underline{s}_2|^2_{\Lambda^{-1}}$$

There remains only the task of maximizing P_E over the remaining free variable β. To maximize P_E, we minimize d^2.

$$d^2 = |\underline{s}_1|^2_{\Lambda^{-1}} + |\underline{s}_2|^2_{\Lambda^{-1}} - 2\text{Re}\{\underline{s}_1^{H}\Lambda^{-1}\underline{s}_2 e^{j\beta}\}$$

Obviously, this expression is minimized by making the last term as negative as possible.

$$d^2_{min} = |\underline{s}_1|^2_{\Lambda^{-1}} + |\underline{s}_2|^2_{\Lambda^{-1}} - 2|\underline{s}_1^{H}\Lambda^{-1}\underline{s}_2|$$

When N snapshots $\underline{z}_n$, n=1,N are available the result is again (III.1) where now

$$(Ad)^2 = \sum_{n=1}^{N} A_n^2 |\underline{s}_1 - e^{j\beta_n} \underline{s}_2|^2_{\Lambda^{-1}}$$

This is minimized term by term over the nuisance parameters β_n to get

$$(Ad)^2 = \sum_{n=1}^{N} A_n^2 \left(|\underline{s}_1|^2_{\Lambda^{-1}} + |\underline{s}_2|^2_{\Lambda^{-1}} - 2|\underline{s}_1^H \Lambda^{-1} \underline{s}_2| \right)$$

which is used in (III.1).

APPENDIX IV
MULTIPLE SIGNALS

The preceding analysis applies virtually unchanged to the multiple signal problem. The hypotheses are

$$H : \underline{z} = Ae^{j\phi_1}\underline{v}(u_1) + \sum_{i=2}^{I} A_i e^{j\phi_1}\underline{v}(u_i) + \underline{\xi} = V(\underline{u})\,\underline{p} + \underline{\xi}$$

$$H' : \underline{z} = Ae^{j\phi'_1}\underline{v}(u'_1) + \sum_{i=2}^{I} A'_i\, e^{j\phi' 1}\underline{v}(u'_i) + \underline{\xi} = V(\underline{u'})\,\underline{p'} + \underline{\xi}$$

where $\underline{u}$ is a vector of angles of arrival and $\underline{p}$ is a vector of complex amplitudes. Because of the multiplicity of parameters, we have changed notation; the parameter values under the two hypotheses are now denoted by unprimed and primed variables. Also, to avoid degeneracies caused by reindexing of parameters, we specify that

$$u_1 < u_2 < \ldots < u_I \text{ and } u'_1 < u'_2 < \ldots < u'_I .$$

Proceeding as before, we conclude that

$$P_E(\underline{p}, \underline{p'}) = \text{erfc}_*\left(\frac{Ad(\underline{p}, \underline{p'})}{\sqrt{2}}\right)$$

where

$$d^2 = \frac{1}{A^2}\,|V\,\underline{p} - V'\,\underline{p'}|^2_{\Lambda^{-1}} = |\underline{v}_1 + \sum_{i=2}^{I} b_i \underline{v}_i - e^{j\gamma}\underline{v'}_1 - \sum_{i=2}^{I} b'_i \underline{v'}_i|^2_{\Lambda^{-1}}$$

$$\triangleq |\underline{v}_1 - e^{j\gamma}\underline{v'}_1 + \tilde{V}\,\underline{b} - \tilde{V}'\,\underline{b'}|^2_{\Lambda^{-1}}$$

This is to be minimized with respect of γ, $\underline{b}$, $\underline{b}'$, and u_i, u_i', i=2, . . . I *
Define

$$W = \Lambda^{-1/2}[\tilde{V} \quad \tilde{V}']$$

$$\underline{a}^T = (-\underline{b}^T, \underline{b}'^T)$$

$$\underline{s}_1 = \Lambda^{-1/2}(\underline{v}_1 - e^{j\gamma}\underline{v}'_1)$$

so that

$$d^2 = |\underline{s}_1 - W\underline{a}|^2$$

which is minimized by choosing $\underline{a} = W^+\underline{s}_1$.

$$d^2 = |(I - WW^+)\underline{s}_1|^2 = |(I - WW^+)\Lambda^{-1/2}(\underline{v}_1 - e^{j\gamma}\underline{v}'_1)|^2$$

$$\triangleq |\underline{w}_1 - e^{j\gamma}\underline{w}'_1|^2 = |\underline{w}_1|^2 + |\underline{w}'_1|^2 - 2\mathrm{Re}\{\underline{w}_1^H \underline{w}'_1 e^{j\gamma}\}$$

$$\min_\gamma d^2 = |\underline{w}_1|^2 + |\underline{w}'_1|^2 - 2|\underline{w}_1^H \underline{w}'_1|$$

$$= (|\underline{w}_1| - |\underline{w}'_1|)^2 + 2|\underline{w}_1|\,|\underline{w}'_1|\left(1 - \frac{|\underline{w}_1^H \underline{w}'_1|}{|\underline{w}_1|\,|\underline{w}'_1|}\right)$$

The vectors $\underline{w}_1$, $\underline{w}'_1$ are the components of $\Lambda^{-1/2}\underline{v}_1$, $\Lambda^{-1/2}\underline{v}'_1$, respectively, which are orthogonal to the space spanned by the columns of W. The distance d is a function of the 2I directions of arrival u_1, . . . u_I, u'_1, . . . u'_I. Maximization over these variables must be done numerically and requires a fine sampling grid. This appears to be impractical for more than one signal.

---

*Without the ordering restriction on the u's, d can be made zero by setting $u_2=u'_1$, $b_2=e^{j\lambda}$, $u'_2=u_1$, $b'_2=1$. This amounts to a reindexing of parameters. Note also that $A'_1=A_1$ in accordance with the discussion in Section II.2.B.

APPENDIX V

PROBABILITY OF ERROR IN CHOOSING BETWEEN TWO GAUSSIAN SIGNALS

The contending hypotheses are

$$H_1 : \underline{z}_n = p_n \underline{v}(u_1) + \underline{\xi}_n$$

$$n = 1, \ldots N$$

$$H_2 : \underline{z}_n = p_n \underline{v}(u_2) + \underline{\xi}_n$$

where $\underline{\xi}_n$ is the usual complex Gaussian noise vector with zero mean and covariance matrix Λ_o, and p_n is a complex Gaussian random variable with statistics

$$E(p_n) = E(p_n^2) = 0, \; E|p_n^2| = MP$$

The observations $\underline{z}_n$ are therefore zero mean complex Gaussian random vectors whose covariance matrix depends on which hypothesis is true.

$$E(\underline{z}_n \underline{z}_n^H | H_1) = \Lambda_1 = \Lambda_o + MP \underline{v}_1 \underline{v}_1^H$$

The log likelihood ratio is

$$\ell = \ell n \frac{P_1 P(\underline{z}_1, \ldots \underline{z}_N / H_1)}{P_2 P(\underline{z}_1, \ldots \underline{z}_N / H_2)} = N \ell n \frac{|\Lambda_2|}{|\Lambda_1|} + \sum_{n=1}^{N} \underline{z}_n^H (\Lambda_2^{-1} - \Lambda_1^{-1}) \underline{z}_n + \ell n \frac{P_1}{P_2}$$

We assume that Λ_o is positive definite so that $\Lambda_o^{1/2}$, $\Lambda_o^{-1/2}$ exist. Then

$$|\Lambda_1| = |\Lambda_o^{1/2} (I + MP \, \underline{w}_1 \underline{w}_1^H) \Lambda_o^{1/2}| = |\Lambda_o| \, |I + MP \, \underline{w}_1 \underline{w}_1^H|$$

where $\underline{w}_1 = \Lambda_o^{-1/2} \underline{v}_1$. The second determinant on the right is readily evaluated

by enumerating its eigenvectors and eigenvalues, with the result

$$|\Lambda_1| = (1 + MP\underline{w}_1^H\underline{w}_1)\ |\Lambda_o|$$

For brevity, we define $q_i = MP\ \underline{w}_i^H\underline{w}_i$.

The inverse Λ_i^{-1} is evaluated using Schur's formula.

$$\Lambda_i^{-1} = \Lambda_o^{-1/2}\left(I - \frac{MP}{1+q_i}\,\underline{w}_i\underline{w}_i^H\right)\Lambda_o^{-1/2}$$

Using these results reduces the log likelihood ratio to

$$\ell = N\,\ell n\,(1 + q_2) - N\,\ell n\,(1 + q_1) + \ell nP_1 - \ell nP_2$$

$$+ \frac{MP}{1 + q_1}\sum |\underline{w}_1^H \Lambda_o^{-1/2}\underline{z}_n|^2 - \frac{MP}{1 + q_2}\sum |\underline{w}_2^H \Lambda_o^{-1/2}\underline{z}_n|^2$$

Define the complex Gaussian random vectors

$$\underline{x}_n = \sqrt{MP}\begin{bmatrix}(1 + q_1)^{-1/2}\,\underline{w}_1^H \\ (1 + q_2)^{-1/2}\,\underline{w}_2^H\end{bmatrix}\Lambda_o^{-1/2}\underline{z}_n$$

and the 2 x 2 matrix

$$J = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

The likelihood ratio test can then be written in the simple form

$$\sum \underline{x}_n^H J\underline{x}_n + N\,\ell n\,\frac{1 + q_2}{1 + q_1} + \ell n\,\frac{P_1}{P_2} \underset{H_2}{\overset{H_1}{\gtrless}} 0$$

The first term is just the difference of two chi-squared variates with 2N degrees of freedom. If the two components of $\underline{x}_n$ were uncorrelated (and therefore statistically independent), then the chi-squared variates would also be independent and the probability of error could be calculated as shown in Appendix VI. Since this is not the case, we seek a linear transformation T of $\underline{x}_n$ which will give independent components while preserving the simplicity of the decision region.

We first compute the covariance matrix of $\underline{x}_n$ under each hypothesis. The results are

$$E\left(\underline{x}_n \underline{x}_n^H / H_1\right) = \begin{bmatrix} q_1 & \left(\frac{1+q_1}{1+q_2}\right)^{1/2} q_{12} \\ \text{Hermitian} & \frac{q_2 + |q_{12}|^2}{1+q_2} \end{bmatrix} \triangleq \begin{bmatrix} c_{11} & c_{12} \\ c_{12}^* & c_{22} \end{bmatrix}$$

$$E\left(\underline{x}_n \underline{x}_n^H / H_2\right) = \begin{bmatrix} \frac{q_1 + |q_{12}|^2}{1+q_1} & \left(\frac{1+q_2}{1+q_1}\right)^{1/2} q_{12} \\ \text{Hermitian} & q_2 \end{bmatrix}$$

where $q_{12} = MP\underline{w}_1^H \underline{w}_2$.

Define the transformed vector.

$$\underline{y}_n = T\,\underline{x}_n = \begin{bmatrix} 1 & a \\ b & c \end{bmatrix} \underline{x}_n$$

The covariance matrix of $\underline{y}_n$ under H_1 is

$$E\left(\underline{y}_n \underline{y}_n^H / H_1\right) = \begin{bmatrix} c_{11} + 2\mathrm{Re}(a^* c_{12}) + |a|^2 c_{22} & c_{11} b^* + c_{12} c^* + c^* a b^* + c_{22} a c^* \\ \text{Hermitian} & |b|^2 c_{11} + 2\mathrm{Re}(c_{12} b c^*) + |c|^2 c_{22} \end{bmatrix}$$

while the test statistic $t = \sum \underline{x}_n^H J \underline{x}_n$ becomes

$$t = \frac{1}{|c-ab|^2} \sum \underline{y}_n^H \begin{bmatrix} |c| - |b| & b^* - c^* a \\ b - ca^* & |a|^2 - 1 \end{bmatrix} \underline{y}_n$$

To preserve the form of the test statistic as a difference of chi-squared variates, we must choose $b=ca^*$. The test statistic then becomes

$$t = \frac{1}{1 - |a|^2} \sum \underline{y}_n^H \begin{bmatrix} 1 & 0 \\ 0 & -|c|^{-2} \end{bmatrix} \underline{y}_n$$

We may as well choose $c=1$ so that

$$t = \frac{1}{1 - |a|^2} \sum \underline{y}_n^H J \underline{y}_n \qquad \text{(V.1)}$$

With these choices, the covariance matrix under H_1 becomes

$$E(\underline{y}_n \underline{y}_n^H / H_1) = \begin{bmatrix} c_{11} + 2\mathrm{Re}(a^* c_{12}) + |a|^2 c_{22} & c_{12}^* (a^2 + \frac{c_{11}+c_{22}}{c_{12}^*} a + \frac{c_{12}}{c_{12}^*}) \\ \text{Hermitian} & |a|^2 c_{11} + 2\mathrm{Re}(a^* c_{12}) + c_{22} \end{bmatrix}$$

The off-diagonal elements are made zero by choosing

$$a = -\frac{1}{\lambda^*} (1 \pm \sqrt{1 - |\lambda|^2}) \qquad \text{(V.2)}$$

where

$$\lambda = \frac{2\, c_{12}}{c_{11}+c_{22}} = \frac{2\, [(1+q_1)(1+q_2)]^{1/2} q_{12}}{q_1 + q_2 + q_1 q_2 + |q_{12}|^2}$$

and the covariance matrix becomes

$$E(\underline{y}_n \underline{y}_n^H / H_1) = \begin{bmatrix} q_1 + 2\left(\frac{1+q_1}{1+q_2}\right)^{1/2} \mathrm{Re}(a^* q_{12}) + |a|^2 \frac{q_2 + |q_{12}|^2}{1+q_2} & 0 \\ 0 & |a|^2 q_1 + 2\left(\frac{1+q_1}{1+q_2}\right)^{1/2} \mathrm{Re}(a^* q_{12}) + \frac{q_2 + |q_{12}|^2}{1+q_2} \end{bmatrix} \quad (V.3)$$

A similar exercise under H_2 leads to the same expression for λ, showing that the transformation works regardless of which hypothesis is true. The covariance under H_2 is

$$E(\underline{y}_n \underline{y}_n^H / H_2) = \begin{bmatrix} |a|^2 q_2 + 2\left(\frac{1+q_2}{1+q_1}\right)^{1/2} \mathrm{Re}(a^* q_{12}) + \frac{q_1 + |q_{12}|^2}{1+q_1} & 0 \\ 0 & q_2 + 2\left(\frac{1+q_2}{1+q_1}\right)^{1/2} \mathrm{Re}(a^* q_{12}) + |a|^2 \frac{q_1 + |q_{12}|^2}{1+q_1} \end{bmatrix} \quad (V.4)$$

We have not yet chosen a sign in the expression (V.2) for a. Since any covariance matrix is nonnegative definite, we have $|c_{12}|^2 \leq c_{11} c_{22}$, and so

$$1 - |\lambda| = 1 - \frac{2|c_{12}|}{c_{11} + c_{22}} > 1 - \frac{2\sqrt{c_{11} c_{22}}}{c_{11} + c_{22}} = \frac{(\sqrt{c_{11}} - \sqrt{c_{22}})^2}{c_{11} + c_{22}} > 0$$

so that $|\lambda| \leq 1$. From a right triangle with hypotenuse 1 and side $|\lambda|$ we have

$$|\lambda| > 1 - \sqrt{1 - |\lambda|^2}$$

which shows that $|a| \leq 1$ provided we choose the negative sign in (V.2). This in turn means that the factor $1 - |a|^2$ is nonnegative, and so the algebraic sign of the test statistic (V.1) is preserved.

We can now compute the probability of error for the test. It is

$$P_E = P_1 \Pr\{t < \delta/H_1\} + (1-P_1)\Pr\{t > \delta/H_2\}$$

where

$$\delta = N \ln \frac{1+q_1}{1+q_2} + \ln \frac{P_2}{P_1}$$

and

$$\Pr(t < \delta/H_1) = \Pr\Big(\sum |y_{n1}|^2 < \sum |y_{n2}|^2 + (1 - |a|^2)\delta \Big)$$

is computed using the results of Appendix VI. The variances of y_{n1} and y_{n2} are obtained from (V.3) under H_1, and from (V.4) under H_2.

The probability of error is, of course, a function of the a priori probability P_1 that H_1 is true. To get the tightest Ziv-Zakai bound we must choose P_1 to maximize P_E. We now show that if the noise is white ($\Lambda_0 = \sigma^2 I$), $P_1 = 1/2$ is the appropriate choice.

When $\Lambda_o = \sigma^2 I$, we have

$$q_1 = q_2 = \frac{MP}{\sigma^2} \triangleq \rho, \quad q_{12} = \rho \underline{v}(u_1)^H \underline{v}(u_2)$$

and the covariance matrices become

$$E(\underline{y}_n \underline{y}_n^H/H_1) = \rho \begin{bmatrix} 1+2\mathrm{Re}(a^* \underline{v}_1^H \underline{v}_2) + |a|^2 \dfrac{1+\rho|\underline{v}_1^H \underline{v}_2|^2}{1+\rho} & 0 \\ 0 & |a|^2+2\mathrm{Re}(a^* \underline{v}_1^H \underline{v}_2) + \dfrac{1+\rho|\underline{v}_1^H \underline{v}_2|^2}{1+\rho} \end{bmatrix} \quad \text{(V.5)}$$

$$E(\underline{y}_n \underline{y}_n^H/H_2) = \rho \begin{bmatrix} |a|^2 + 2\mathrm{Re}(a^* \underline{v}_1 \underline{v}_2) + \dfrac{1+\rho|\underline{v}_1^H \underline{v}_2|^2}{1+\rho} & 0 \\ 0 & |a|^2 + 2\mathrm{Re}(a^* \underline{v}_1 \underline{v}_2) + \dfrac{1+\rho|\underline{v}_1^H \underline{v}_2|^2}{1+\rho} \end{bmatrix} \quad \text{(V.6)}$$

We observe that the variates y_{n1}, y_{n2} simply change their statistical identities under an hypothesis change. Thus $U = \Sigma|y_{n1}|^2$, $V = \Sigma|y_{n2}|^2$ also reverse roles. It follows that in any conditional probability expression involving these variable, we can change hypotheses by interchanging U and V, e.g.

$$\Pr(U > V + \delta/H_1) = \Pr(V > U + \delta/H_2)$$

Since for white noise $q_1 = q_2$, it follows that

$$\delta = \ell n \frac{1-P_1}{P_1}$$

It will be convenient to minimize P_E with respect to δ rather than P_1. Thus

$$P_1 = \frac{1}{1+e^{\delta}}$$

$$1-P_1 = \frac{1}{1+e^{-\delta}}$$

The probability of error is

$$P_E(\delta) = \frac{1}{1+e^{\delta}} \Pr\{U < V + b\delta/H_1\} + \frac{1}{1+e^{-\delta}} \Pr\{U > V + b\delta/H_2\}$$

where for brevity we have put $b = 1 - |a|^2$. We first show that P_E is a symmetric function of δ.

$$P_E(-\delta) = \frac{1}{1+e^{-\delta}} \Pr\{U < V - b\delta/H_1\} + \frac{1}{1+e^{\delta}} \Pr\{U > V - b\delta/H_2\}$$

$$= \frac{1}{1+e^{-\delta}} \Pr\{U > V + b\delta/H_2\} + \frac{1}{1+e^{\delta}} \Pr\{U < V + b\delta/H_1\} = P_E(\delta)$$

This result, as well as those to follow, uses the fact that the distribution function of U-V is continuous, so that the distinction between $<$, $\leq$ can be ignored.

The derivative of P_E is

$$\frac{dP_E}{d\delta} = \frac{1}{(1+e^{\delta})(1+e^{-\delta})} \left[\Pr\{U > V + b\delta/H_2\} - \Pr\{U < V + b\delta/H_1\}\right] \tag{V.7}$$

$$+ b\left[\frac{1}{1+e^{\delta}} \Pr\{U = V + b\delta/H_1\} - \frac{1}{1+e^{-\delta}} \Pr\{U = V + b\delta/H_2\}\right]$$

It is readily verified that this expression vanishes when $\delta = 0$, and that

$$P_E(0) = \Pr\{U < V/H_1\}$$

Furthermore, when $\delta > 0$, the first bracket in (V.7) is negative. We now show that the second bracket is also negative.

The pdf of U under H_1 is

$$p(U) = \frac{1}{(N-1)!} \frac{U^{N-1}}{(2\sigma_U^2)^N} e^{-\frac{U}{2\sigma_U^2}} \qquad U > 0$$

and similarly for V. Define for brevity

$$x = \frac{U}{2\sigma_U^2}, \quad y = \frac{V}{2\sigma_U^2}, \quad r = \frac{\sigma_V^2}{\sigma_U^2}, \quad c = \frac{b\delta}{2\sigma_U^2}$$

the variances being those under H_1 given by (V.5). Note that

$$\sigma_U^2 - \sigma_V^2 = (1 - |a|^2)\left(1 - \frac{1+p|\underline{v}_1^H\underline{v}_2|^2}{1+p}\right) > 0$$

since $|a| < 1$, $|\underline{v}_1^H\underline{v}_2| < 1$ and therefore $0 < r < 1$.

Then

$$\Pr\{U = V + b\delta/H_1\} = \int_0^\infty p_V(v)p_U(v+b\delta)dv$$

$$= \frac{1}{2\sigma_U^2} \frac{e^{-c}}{((N-1)!)^2 r^N} \int_0^\infty y^{N-1}(y+c)^{N-1} e^{-\frac{r+1}{r}y} dy$$

and

$$\Pr\{U = V + b\delta/H_2\} = \Pr\{V = U + b\delta/H_1\}$$

$$= \int_0^\infty p_U(u)p_V(u+b\delta)du$$

$$= \frac{1}{2\sigma_U^2} \frac{e^{-c/r}}{((N-1)!)^2} \int_0^\infty x^{N-1}(x+c)^{N-1} e^{-\frac{r+1}{r}x} dx$$

$$= e^{-c\frac{1-r}{r}} \Pr\{U = V + b\delta/H_1\}$$

The second term can then be written as constant $(\frac{1+e^{-\delta}}{1+e^{\delta}} e^{-c\frac{1-r}{r}} - 1)$, which is negative for $w > 0$, $r \leq 1$.

We have shown that the derivative of P_E vanishes at $\delta=0$ and is negative for $\delta>0$. It follows from the symmetry of P_E that its unique maximum occurs at $\delta=0 (P_1 = 1/2)$.

APPENDIX VI

PROBABILITY THAT THE DIFFERENCE OF TWO X^2 VARIATES EXCEEDS A THRESHOLD

Let u and v be X^2 variates with 2N degrees of freedom, having variances (per degree of freedom) σ_u^2, σ_v^2 respectively. The pdf of u is

$$p(u) = \frac{1}{(2\sigma_u^2)^N} \frac{u^{N-1}}{(N-1)!} e^{-\frac{u}{2\sigma_u^2}} \qquad u > 0 \qquad \text{(VI.1)}$$

Then for $\delta > 0$

$$\Pr(u > v + \delta) = \int_0^\infty p(v) \int_{v+\delta}^\infty p(u)dudv \qquad \text{(VI.2)}$$

Using (VI.1) and (VI.2), we obtain

$$\Pr(u > v + \delta) = \frac{1}{((N-1)!)^2} \int_0^\infty v^{N-1} e^{-v} \int_{rv+c}^\infty u^{N-1} e^{-u} dudv \qquad \text{(VI.3)}$$

where $r = \frac{\sigma_v^2}{\sigma_u^2}$, $c = \frac{\delta}{2\sigma_u^2}$. Using the result

$$\int_a^\infty u^{N-1} e^{-u} du = (N-1)! \sum_{n=0}^{N-1} \frac{a^n}{n!} e^{-a} \qquad \text{(VI.4)}$$

in (VI.3), we expand $(rv+c)^n$ in a binomial series, use (VI.4) again, change a summation index, and obtain eventually (still for $\delta > 0$)

$$\Pr(u > v + \delta) = \sum_{m=0}^{N-1} \binom{N+m-1}{m} \frac{r^m}{(1+r)^{N+m}} \left(\sum_{n=0}^{N-1-m} \frac{c^n}{n!} e^{-c} \right) \qquad \text{(VI.5)}$$

which is the desired result.

If $\delta < 0$, (VI.5) can be used to get the desired result.

$$\Pr(u > v + \delta) = \Pr(v < u - \delta) = 1 - \Pr(v > u - \delta)$$

The last probability can be computed from (VI.5) since $-\delta \geq 0$.

When $\delta = 0$, this reduces to

$$\Pr(u > v) = \sum_{m=0}^{N-1} \binom{N+m-1}{m} \frac{r^m}{(1+r)^{N+m}}$$

which can be put in more convenient form.

$$\Pr(u > v) = \frac{1}{(1+r)^{2N-1}} \sum_{m=0}^{N-1} \binom{N+m-1}{m} r^m (1+r)^{N-1-m}$$

Expand the last factor in a binomial series.

$$\Pr(u > v) = \frac{1}{(1+r)^{2N-1}} \sum_{m=0}^{N-1} \binom{N+m-1}{m} r^m \sum_{n=0}^{N-1-m} \binom{N-1-m}{n} r^{N-1-m-n}$$

Interchange the order of summation

$$\Pr(u > r) = \frac{1}{(1+r)^{2N-1}} \sum_{n=0}^{N-1} r^{N-1-n} \sum_{m=0}^{N-1-n} \binom{N+m-1}{m} \binom{N-m-1}{n}$$

Replace n by N-1-n

$$\Pr(u > v) = \frac{1}{(1+r)^{2N-1}} \sum_{n=0}^{N-1} r^n \sum_{m=0}^{n} \binom{N+m-1}{m} \binom{N-m-1}{n-m}$$

Finally, using an identity involving binomial coefficients (Feller [13], p.62), we obtain

$$\Pr(u > v) = \sum_{n=0}^{N-1} \binom{2N-1}{n} \left(\frac{r}{1+r}\right)^n \left(\frac{1}{1+r}\right)^{2N-1-n}$$

This is just the probability of less than N successes in 2N-1 trials with probability of success $\left(\frac{r}{1+r}\right)$. This form has been given previously by Kelly [14].

APPENDIX VII

LINEAR DEPENDENCE AND THE ZIV-ZAKAI BOUND

A development of the Ziv-Zakai bound on accuracy in estimating direction u_1 in the presence of multiple signals is given in Appendix IV. It is shown that the probability of error is

$$P_E = \text{erfc}_*\left(\frac{Ad}{\sqrt{2}}\right)$$

where, for two signals,

$$d^2 = |(I - WW^+)\, \underline{s}_1|^2$$

$$\underline{s}_1 = \Lambda^{-1/2}\,[\underline{v}(u_1) - e^{j\gamma}\underline{v}(u_1')] \triangleq \underline{\tilde{v}}(u_1) - e^{j\gamma}\underline{\tilde{v}}\,(u_1')$$

$$W = \Lambda^{-1/2}[\underline{v}(u_2)\ ,\ \underline{v}(u'_2)] \triangleq [\underline{\tilde{v}}(u_2),\ \underline{\tilde{v}}(u_2')]$$

The vector $(I - WW^+)\, \underline{s}_1$ is just the component of $\underline{s}_1$ orthogonal to the subspace spanned by the columns of W. Suppose that $\underline{\tilde{v}}(u_1)$, $\underline{\tilde{v}}(u_2)$, $\underline{\tilde{v}}(u_3)$ are linearly dependent direction vectors* and set $u_1' = u_2$, $u_2' = u_3$. Then $\underline{s}_1$ is a linear combination of the columns of W and so d=0. This leads to the two-point bound

$$\tfrac{1}{2}\left| e^2(u_1, A) + e^2(u_2, A)\right| > \frac{(u_1 - u_2)^2}{4}$$

since $P_e = \frac{1}{2}$. We are assured that the error at either u_1 or u_2 (whichever is worse) exceeds the bound, no matter how large the signal-to-noise ratio may be.

Suppose instead that $\underline{\tilde{v}}(u_1)$, $\underline{\tilde{v}}(u_2)$, $\underline{\tilde{v}}(u_3)$ are "almost" linearly dependent in the sense that the determinant $|\tilde{V}^H\tilde{V}|$ is nearly zero. The component of s_1,

---

*Since $\underline{\tilde{v}}_i = \Lambda^{-1/2}\underline{v}_i$ and $\Lambda^{-1/2}$ is nonsingular, the $\underline{\tilde{v}}_i$ are linearly independent if and only if the $\underline{v}_i$ are.

or equivalently $\tilde{\underline{v}}_1$, orthogonal to the space spanned by $\tilde{\underline{v}}_2$, $\tilde{\underline{v}}_3$ has squared length

$$d^2 = \min_{a,b} |\tilde{\underline{v}}_1 - a\tilde{\underline{v}}_2 - b\tilde{\underline{v}}_3|^2$$

$$= \min_{\underline{w}} \underline{w}^H \tilde{V}^H \tilde{V} \underline{w} \quad \text{subject to } \underline{e}^H \underline{w} = 1$$

where $\underline{e}^H = (1, 0, 0)$. The solution to this familiar problem is

$$d^2_{min} = \frac{1}{\underline{e}^H (\tilde{V}^H \tilde{V})^{-1} \underline{e}} = \frac{|\tilde{V}^H \tilde{V}|}{|\tilde{V}^H \tilde{V}|}$$

where $\tilde{V} = (\tilde{\underline{v}}_2 \ \tilde{\underline{v}}_3)$. The denominator is a measure of the linear dependence of $\tilde{\underline{v}}_2$ and $\tilde{\underline{v}}_3$, which does not affect the accuracy of estimating u_1.

A contour plot of $1 - d^2$ as a function of $u_2 - u_1$ and $u_3 - u_2$ for the QUICK LOOK array is shown in Fig. VII.1*. Since d^2 is not symmetric in these two variables, the entire region $0 < u_3 - u_2 < 2 - (u_2 - u_1) < 2$ must be plotted. The result is very similar to a contour plot of $|\tilde{V}^H \tilde{V}|$ (Fig. 3.6) except that the horizontal ridges which represent near linear dependence of $\underline{v}_2$ and $\underline{v}_3$, are missing. This is due to the denominator factor. If we compute d^2 for estimating u_3, the vertical ridges disappear; for u_2, the slanting ridges. Since we are ultimately interested in all three accuracies, the Gramian determinant $|V^H V|$ is most appropriate.

Since $d > 0$, increasing S/N ratio will eventually make P_E small. Comparing two situations, one with minimum distance d_1, the other with minimum distance d_2, the second will require $20 \log d_2/d_1$ dB additional S/N ratio for equal accuracy.

We have shown that when $\tilde{\underline{v}}_1$ is nearly a linear combination of $\tilde{\underline{v}}_2$ and $\tilde{\underline{v}}_3$ the Ziv-Zakai bound predicts large estimation errors in trying to distinguish between a pair of signals from directions u_1, u_2 and a pair from u_2, u_3.

---

*For white noise, i.e., $\Lambda_0 = \sigma^2 I$.

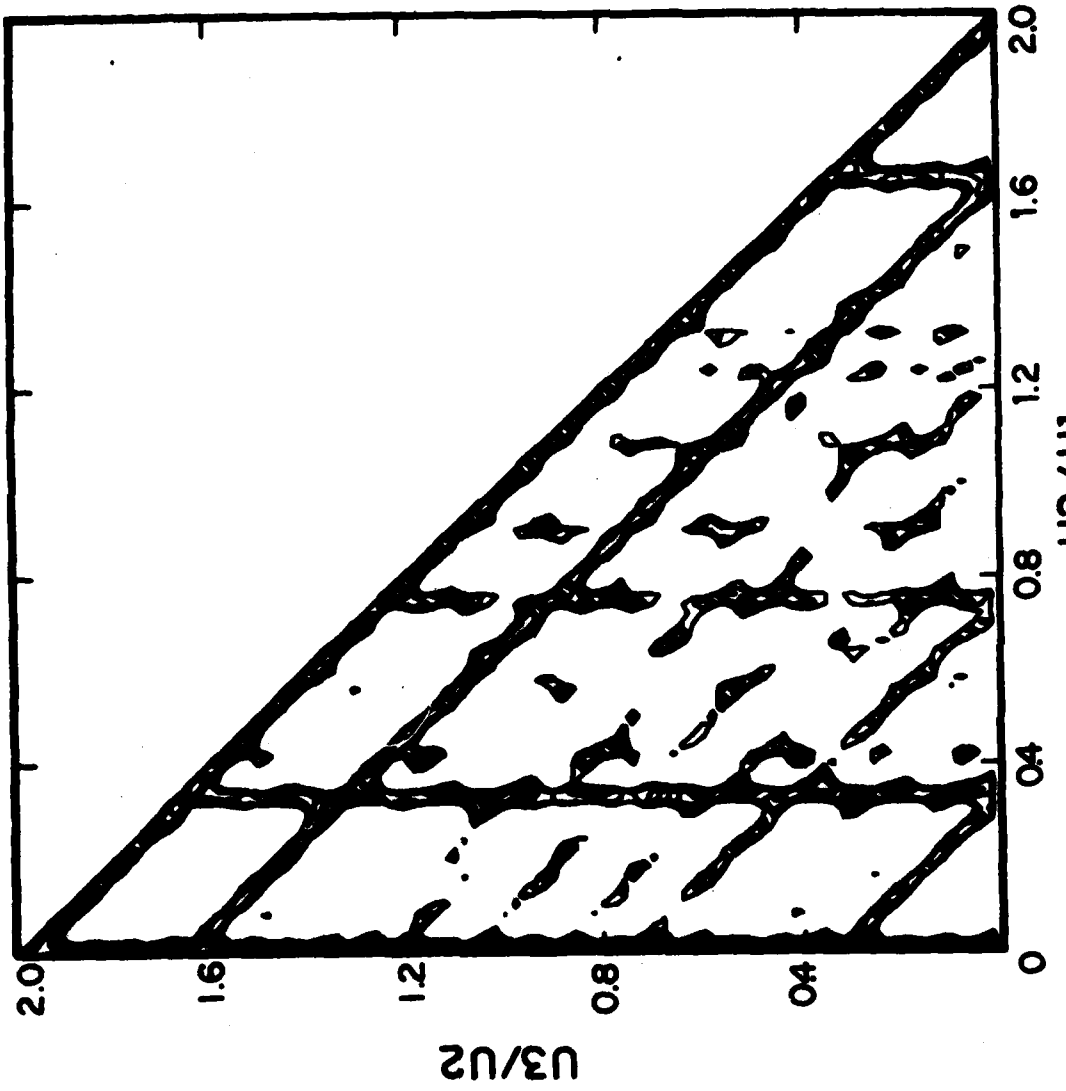


Fig. VII.1. Contour plot of $1 - d^2_{min}$ for QUICK-LOOK array.

APPENDIX VIII
QUICK LOOK PROCESSING MODIFICATION

Phase differences are measured over baselines of 2.5, 5,5, and 12.1 wavelengths (at the highest operating frequency f_h) . Without noise, an emitter at frequency f and direction u will produce the results

$$\begin{aligned} \phi_{2.5} &= 2.5\, f_r\, u - M \\ \phi_{5.5} &= 5.5\, f_r\, u - N \\ \phi_{12.1} &= 12.1\, f_r\, u - L \end{aligned} \qquad \text{(VIII.1)}$$

where the phases are expressed in cycles (i.e., fractions of a complete revolution), f_r is the frequency ratio f/f_H and L, M, N are integers. From these, we derive

$$\frac{\phi_{2.5} + M}{\phi_{5.5} + N} = \frac{\phi_{5.5} + N}{\phi_{12.1} + L} = \frac{5}{11}$$

or

$$\begin{aligned} 11\,\phi_{2.5} - 5\,\phi_{5.5} &= 5N - 11\,M = I_1 \\ 11\,\phi_{5.5} - 5\,\phi_{12.1} &= 5L - 11\,N = I_2 \end{aligned} \qquad \text{(VIII.2)}$$

In reality, of course, the quantities on the left-hand side will not be integers because of noise and interference. However, if these effects are small, they will be nearly integers, and by rounding them to the nearest integers, correct results will be obtained. As the interference becomes larger, the rounded values will be incorrect. Since 5 and 11 are relatively prime, equations (VIII.2) <u>always</u> have a solution, i.e., for any integers I_1, I_2, there exist integers L, M, N such that the equations are satisfied. However, most such solutions can be discarded as physically impossible. Since $-1 < u < 1$, the first equation (VIII.1) implies

$$-2.5\, f_r < \phi + M < 2.5\, f_r \quad .$$

Since

$$0 < \phi < 1 \quad ,$$

we have

$$-2.5\, f_r - 1 < -2.5\, f_r - \phi < M < 2.5\, f_r - \phi < 2.5\, f_r \quad .$$

Since M is an integer,

$$\begin{aligned} -[2.5\, f_r] &< M < [2.5\, f_r] \\ -[5.5\, f_r] &< N < [5.5\, f_r] \\ -[12.1\, f_r] &< L < [12.1\, f_r] \end{aligned} \qquad \text{(VIII.3)}$$

where [x] denotes the largest integer < x.

Solutions which do not satisfy these constraints can be discarded.

The equations (VIII.1) are solved as follows: the first equation is solved for N modulo 11. Since $11M = 0 \bmod 11$, $5N_1 \equiv I_1 \equiv -10\, I_1 \bmod 11$, so

$$N_1 \equiv -2I_1 \bmod 11 \qquad \text{(VIII.4)}$$

A similar procedure applied to the second equation yields

$$N_2 \equiv -I_2 \bmod 5 \qquad \text{(VIII.5)}$$

The physical constraints (VIII.3) require (for $f_r = 1$) that $-6 < N < 5$. Equation (VIII.4) can be rewritten in the equivalent form

$$N_1 \equiv (28 - 2I_1) \bmod 11 - 6 \qquad \text{(VIII.4')}$$

where now $-6 < N_1 < 4$. Similarly, (VIII.5) can be written

$$N_2 \equiv (6 - I_2) \bmod 5 - 1 \qquad \text{(VIII.5')}$$

where $-1 < N_2 < 3$. It is now necessary to check whether or not N_1 and N_2 match within the interval [-6, 5]. Figure VIII.1 shows that matching N_1 with N_2, N_2+5, N_2-5 covers all possibilities but one, namely that $N_1=5$. Thus, if no match occurs after three comparisons, we must compare N_1+11 with N_2+5. This is the required modification to the QUICK LOOK processing algorithm. A block diagram of the modified processing algorithm is shown in Fig. VIII.2.

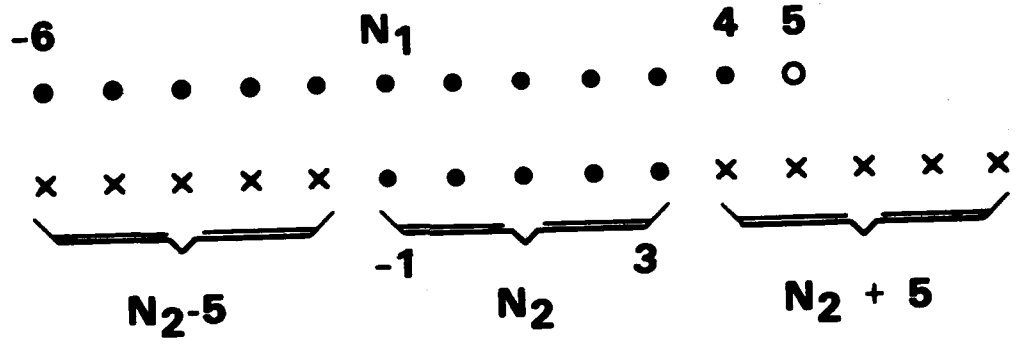


Fig. VIII.I. Matching N_1 and N_2 in the QUICK-LOOK phase ambiguity resolution algorithm.

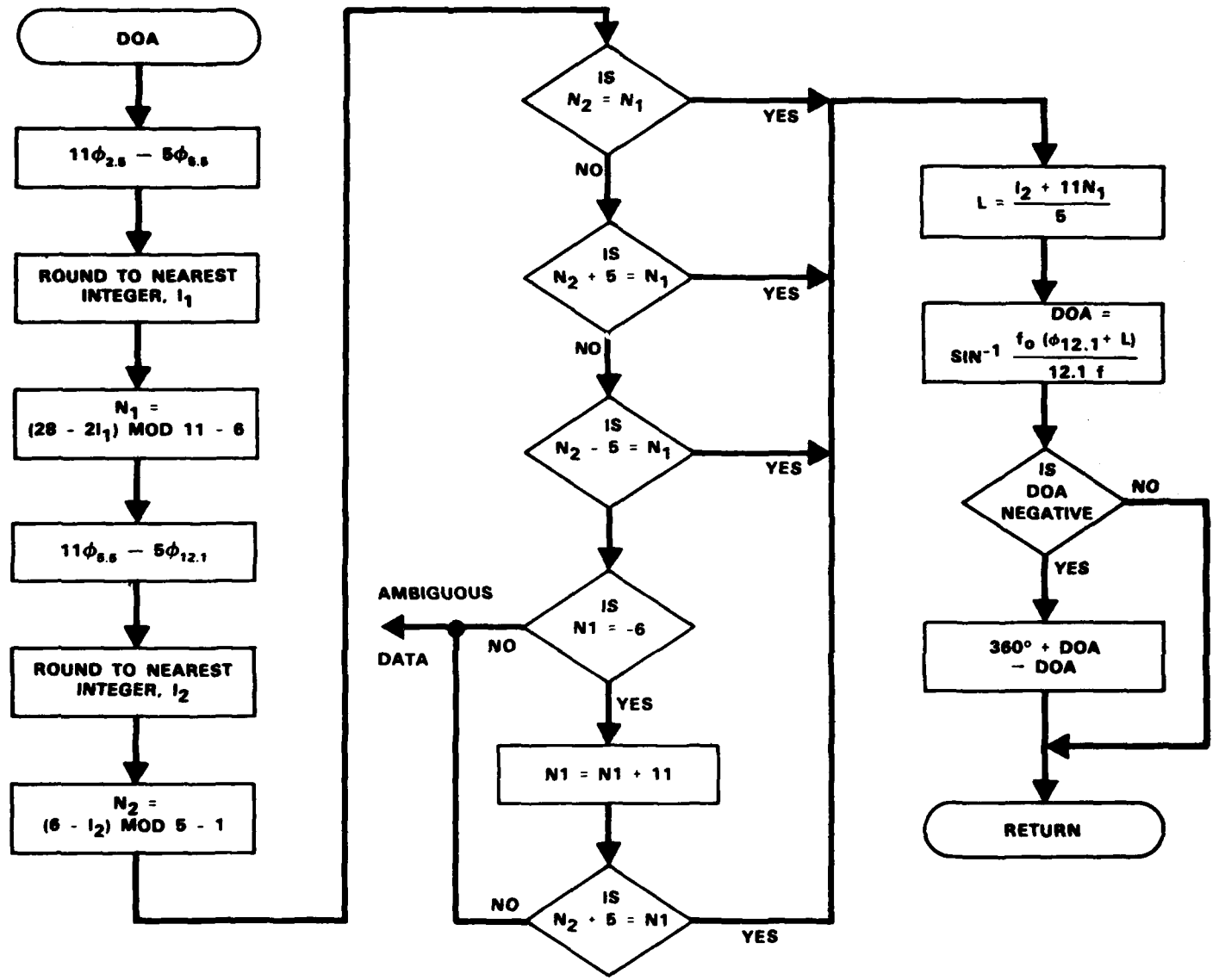


Fig. VIII.2. Modified QUICK-LOOK processing algorithm.

REFERENCES

1. H. L. Van Trees, Detection Estimation and Modulation Theory, Part I, Wiley, New York (1968).

2. R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York (1960).

3. J. E. Evans, J. R. Johnson, and D. F. Sun, "Application of Advanced Signal Processing Techniques to Angle Estimation in ATC Navigation and Surveillance Systems," Technical Report 582, Lincoln Laboratory, M.I.T. (23 June 1982), DTIC-AD-A11830610.

4. M. Athans, F. C. Schweppe, "Gradient Matrices and Matrix Calculations," Technical Note 1965-53, Lincoln Laboratory, M.I.T. (17 November 1965), DDC-624426.

5. D. F. DeLong, "A New Algorithm for Adaptive Radar Signal Processing," Technical Note 1979-72, Lincoln Laboratory, M.I.T. (31 December 1979), DDC-AD-A082856/6.

6. T. W. Miller, S. D. Huber, "Antenna Array Techniques For Fractional Beamwidth Angular Resolution of Multiple Emitters," presented at 1980 Adaptive Antenna Conference.

7. J. Ziv, M. Zakai, "Some Lower Bounds on Signal Parameter Estimation," IEEE Trans. Info. Theory, Vol. IT-15, No. 2 (1969), pp. 386-391.

8. R. S. Orr, R. D. Yates, "On the Estimation of Arrival Time of Pulse Signals in Gaussian Noise," Technical Note 1974-13, Lincoln Laboratory, M.I.T. (26 August 1974), PB-236768/8.

9. M. Wax, J. Ziv, "Improved Bounds on the Local Mean-Square Error and the Bias of Parameter Estimators," IEEE Trans. Info. Theory, Vol. IT-23, pp. 529-530 (July 1977).

10. R. Schmidt, "Multiple Emitter Location and Signal Parameter Estimation," Proc. RADC Spectrum Estimation Workshop, Rome, New York (1979).

11. S. W. Lang, "Spectral Estimation for Sensor Arrays," PhD Thesis, M.I.T. (August 1981).

12. J. S. Lin, N. A. Malik, "A New Algorithm for Two-Dimensional Maximum Entropy Power Spectrum Estimation," IEEE Trans. ASSP, Vol. ASSP-29, pp. 401-413 (June 1981).

13. W. Feller, An Introduction to Probability Theory and Its Applications, Vol. I, Wiley, New York (1957).

14. E. J. Kelly, "Finite-Sum for Signal Detection Probabilities," Technical Report 566, Lincoln Laboratory, M.I.T. (20 May 1981), UTIC-AD-A102143.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ESD-TR-83-006	2. GOVT ACCESSION NO. AD-A128 924	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Multiple Signal Direction Finding With Thinned Linear Arrays		5. TYPE OF REPORT & PERIOD COVERED Project Report 6. PERFORMING ORG. REPORT NUMBER Project Report TST-68
7. AUTHOR(s) Darrol F. DeLong		8. CONTRACT OR GRANT NUMBER(s) F19628-80-C-0002
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lincoln Laboratory, M.I.T. P.O. Box 73 Lexington, MA 02173-0073		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element No. 63250F Project No. 649L
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Systems Command, USAF Andrews AFB Washington, DC 20331		12. REPORT DATE 13 April 1983 13. NUMBER OF PAGES 114
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Electronic Systems Division Hanscom AFB, MA 01731		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

None

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cramer-Rao bound	emitters	signal-to-noise ratios
Ziv-Zakai bounds	direction-finding performance	linear arrays

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The evaluation of the direction-finding capabilities of non-uniform arrays is approached via algorithm-independent lower bounds on achievable angle estimation errors. Two classes of bounds are considered. The first, known as the Cramer-Rao bound, applies only to unbiased estimates. Compact analytical expressions for these bounds are developed which are applicable to very general direction-finding problems, including an arbitrary number of emitters.

DD FORM 1 Jan 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT (Continued)

It is well-known that Cramer-Rao bounds are overly optimistic at low signal-to-noise ratios. As this ratio decreases, a point is reached at which estimation accuracy decreases abruptly. Another class of bounds, known as Ziv-Zakai bounds, provide information about the location of this threshold point. Their study suggests that poor direction-finding performance occurs in situations where the emitter direction vectors are part of a set which is nearly linearly dependent.

Such linear dependence does not occur in the case of uniformly spaced linear arrays (without getting lobes). However, it does occur when elements are removed from such arrays. A systematic test is developed to test a given array geometry for this condition.

Finally, direction-finding performance of the QUICK LOOK array is evaluated via both bounds and simulation.